Some Interesting Consequences of a Hyperbolic Inequality

Frank Burk, Department of Mathematics, CSUC, Chico, CA

Upon first glance, we notice how similar

$$\tanh t < t < \sinh t \qquad (t > 0) \tag{1}$$

is to the familiar inequality

$$\sin t < t < \tan t \qquad (t > 0), \tag{2}$$

used to establish that $\lim_{t\to 0}(\sin t)/t = 1$. We usually convince students of the 'validity' of (2) by comparing the shaded areas in Figure 1.

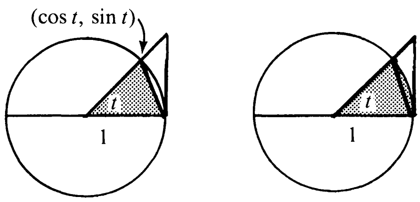

Figure 1.

The same approach (Figure 2), replacing the unit circle $x^2 + y^2 = 1$ with the unit hyperbola $x^2 - y^2 = 1$, yields (1).

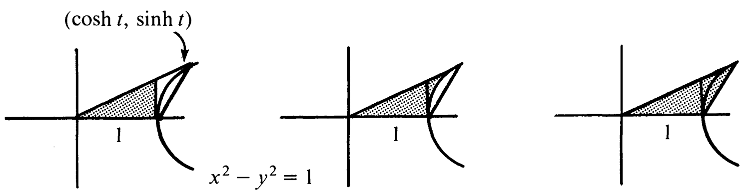

Figure 2.

Although these geometrical arguments do not constitute proofs, they strongly suggest (see Figure 3) the following possible inequalities:

$$\frac{\sinh t}{\sqrt{\cosh^2 t + \sinh^2 t}} < \tanh t < t < \sinh t < \sinh t \cosh t \qquad (t > 0). \tag{3}$$

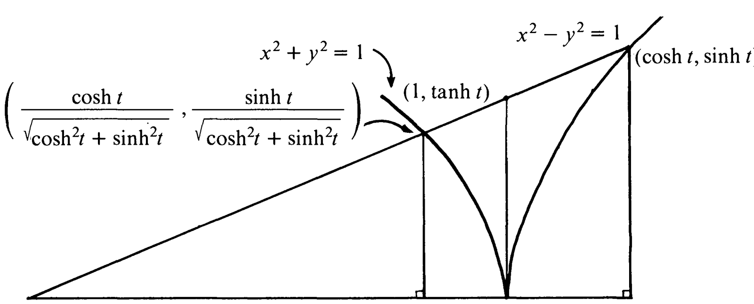

Figure 3.

Now replace t with $\ln\sqrt{b/a}$ $(0 < a < b)$ in (3). After taking reciprocals and simplifying, we get

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} < \sqrt{ab} < \frac{b-a}{\ln b - \ln a} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}} \qquad (0 < a < b), \tag{4}$$

relating the harmonic, geometric, logarithmic, arithmetic, and root-mean-square means.

For a rigorous proof of (1), observe that

$$\cosh x - 1 \equiv \left(\frac{e^{x/2} - e^{-x/2}}{2} \right)^2 > 0 \qquad \text{for} \quad x > 0.$$

Therefore,

$$\frac{1}{\cosh^2 x} < 1 < \cosh x \qquad \text{for} \quad x > 0,$$

and (1) follows by integrating the preceding inequality from $x = 0$ to $x = t$. The other inequalities of (3) are obvious.

Trees and Tennis Rankings

Curtis Cooper, Central Missouri State University, Warrensburg, MO

Everyone is familiar with the usual kind of "elimination" tennis tournament in which losers drop out and winners continue to play until one undefeated player remains. Such a tournament is nicely described by a *tree* whose vertices $1, 2, \ldots, n$ represent the n players and whose edges represent the matches that were played—the direction of each edge pointing toward the loser. In the tournament of Figure 1, for example, player 4 won by beating player 6 (who beat 11 and 1), player 10, and player 3 (who beat 8, 2, and 5). In addition, player 8 beat player 9 (who beat 7 and 12).

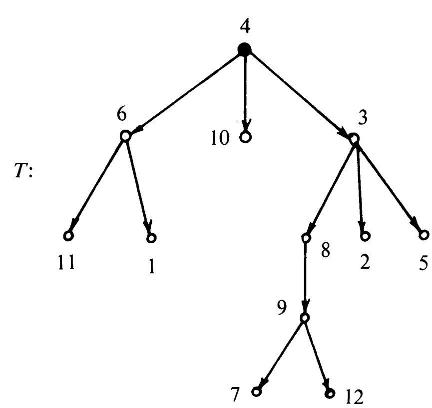

Figure 1.

Suppose we wanted to rank the players of this tournament, subject to the rule: *for each edge of T, player i precedes player j if and only if i beats j.* There is clearly no unique way to do so since T does not contain enough information for a full account