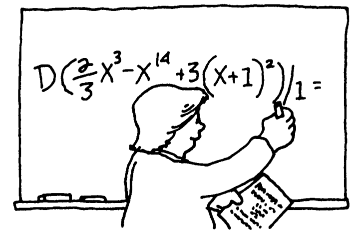

EDITOR

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A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Doughnut Slicing

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In his note "Cylinder and Cone Cutting" [*CMJ* 28:2 (March 1997) 122–123], Michael Cullen points out that, although students are aware that conic sections are formed when a cone is cut by various planes, it is difficult to demonstrate this fact to students unfamiliar with three-dimensional geometry. He overcomes this dilemma by *rotating the cone* through various angles while *keeping the plane fixed*, thereby demonstrating that the standard equations of the conic sections result.

In the following note (submitted before Cullen's appeared), I employ the same rotation of axes technique to investigate a drawing that I ran across depicting the intersection of a plane through the center of a torus and tangent to the surface, as portrayed in Figure 1. The drawing indicated that the intersection consists of two overlapping circles, which at first I found difficult to believe. But with a little analytic geometry I found that this is indeed the case.

Can you picture the intersection?

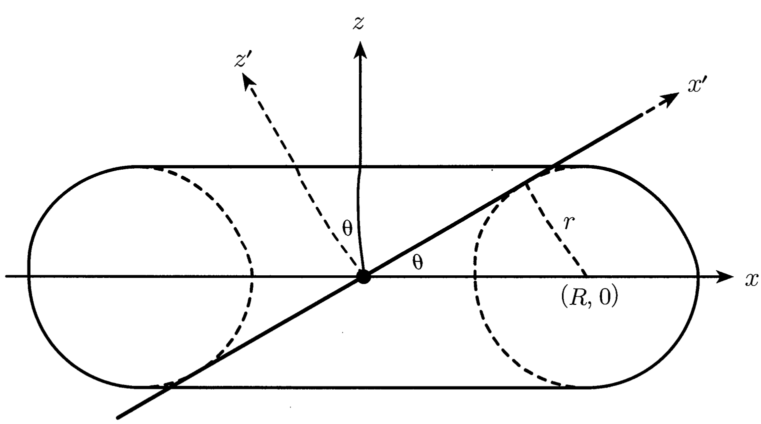

Figure 1. Side view of torus and tangent plane.

To begin, let the torus be produced by revolving about the z-axis the circle

$$(x - R)^2 + z^2 = r^2, \quad r < R,$$

in the xz-plane. The equation of the surface of revolution is then

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2.$$

Let the plane through the center of the torus and tangent to its surface be

$$z = x \tan\theta \quad \text{where } \sin\theta = \frac{r}{R}.$$

Note that this plane contains the y-axis.

Now we rotate the coordinate axes about the y-axis by θ, so that the tangent plane is the $x'y'$-plane. Figure 1 shows the torus and its tangent plane as seen from the negative $y = y'$-axis. The equation of the torus in the rotated coordinate system is

$$\left(\sqrt{(x' \cos\theta - z' \sin\theta)^2 + y'^2} - R\right)^2 + (x' \sin\theta + z' \cos\theta)^2 = r^2.$$

The curve of intersection with the tangent plane is found by simply setting $z' = 0$ in this equation:

$$x'^2 + y'^2 - 2R\sqrt{x'^2 \cos^2\theta + y'^2} + R^2 = r^2.$$

To recognize that this describes a pair of circles in the $x'y'$-plane requires a bit of algebra. Rearranging and squaring both sides,

$$\left(x'^2 + y'^2\right)^2 + 2\left(R^2 - r^2\right)\left(x'^2 + y'^2\right) + \left(R^2 - r^2\right)^2 = 4R^2\left(x'^2 \cos^2\theta + y'^2\right).$$

Subtracting $4(R^2 - r^2)(x'^2 + y'^2)$ from both sides and using $\sin\theta = r/R$ then yields

$$\left(x'^2 + y'^2\right)^2 - 2\left(R^2 - r^2\right)\left(x'^2 + y'^2\right) + \left(R^2 - r^2\right)^2 = 4r^2 y'^2,$$

or

$$\left[x'^2 + y'^2 - \left(R^2 - r^2\right)\right]^2 - (2ry')^2 = 0.$$

This factors as

$$\left[x'^2 + y'^2 - 2ry' + r^2 - R^2\right]\left[x'^2 + y'^2 + 2ry' + r^2 - R^2\right] = 0,$$

or

$$\left[x'^2 + (y' - r)^2 - R^2\right]\left[x'^2 + (y' + r)^2 - R^2\right] = 0.$$

Hence the intersection is two circles of radius R whose centers are on the y'-axis a distance $2r$ units apart. Figure 2 shows the view from the positive z'-axis, with the intersection of the plane and torus emphasized.

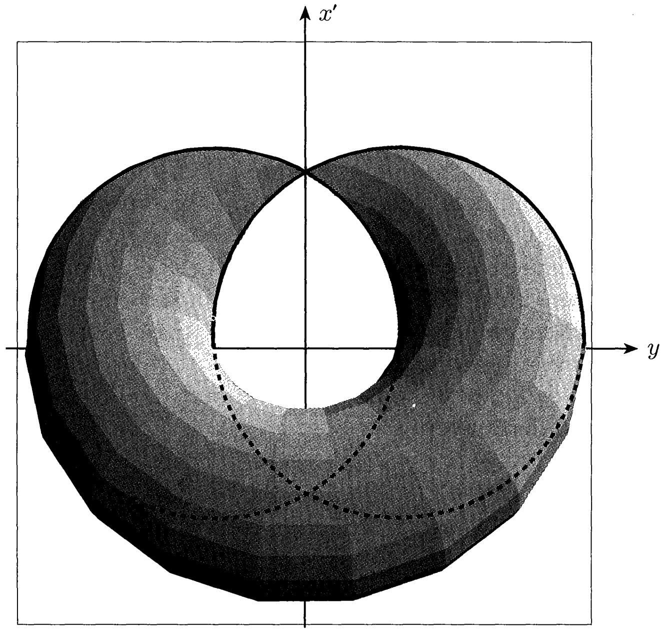

Figure 2

This example, as well as those in Cullen's note, could be used to introduce students to the subtleties of three-dimensional geometry. After one of the examples is discussed in class, the others would make valuable homework exercises.

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Counterexamples to a Weakened Version of the Two-Variable Second Derivative Test

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Most calculus textbooks carefully state the second derivative tests for relative maximum/minimum in one and two dimensions. For example, the presentation in H. Anton's *Calculus* [4th ed., Wiley, New York, 1992] is representative; his single-variable version is found on page 253:

> Suppose f is twice differentiable at a stationary point x_0.
>
> a. If $f''(x_0) > 0$, then f has a relative minimum at x_0.
> b. If $f''(x_0) < 0$, then f has a relative maximum at x_0.

His two-variable version is found on page 1057 (emphasis added):

> Let f be a function of two variables with *continuous second order partial derivatives in some circle centered at a critical point* (x_0, y_0), and let

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$