Now (surprise!) this algorithm is just as efficient. It produces quadratic convergence to $\sqrt[3]{2}$, because (2) happens to produce outputs identical to those obtained by applying Newton's method to solve $x^3 - 2 = 0$. If we start with an input of $5/4$, an easily discovered quotient of small integers whose cube is quite close to 2, then we successively get the approximants $5/4$, $63/50$, and $375047/297675$, the last of which is $1.259921055\ldots$ and thus agrees with $\sqrt[3]{2}$ to eight decimal places.

The pedagogical value of this is that the instructor can follow it by challenging students to figure out how to get quadratic convergence to fourth roots, fifth roots, etc. by extending this line of thought to higher dimensions unrecognized by the Greeks. With the instructor's encouragement, some students may dare to venture into the "fourth dimension" and see that to approach $\sqrt[4]{2}$, for example, they should make an approximation x and take their next approximant y to be

$$y = \frac{1}{4}\left(x + x + x + \frac{2}{x^3}\right). \tag{3}$$

Students may go on from (3) to discover, analogously, how to get the n-th root of an arbitrary positive number (and thereby effortlessly obtain the ability to handle rational roots as well, since $a^{m/n}$ is the n-th root of a^m).

Virtually any reasonable guess will yield an algorithm leading to convergence to a desired root, but the rate of convergence for uninspired guesses can be agonizingly slow. If, for example, in approximating $\sqrt[3]{2}$ one should guess that the output y should be simply the average of x and $2/x^2$, then an initial input of $5/4$ in the resulting algorithm will require about twenty iterations to get the precision obtained by two applications of formula (2).

Playing with such algorithms will lead students to realize that a mathematical task may often be done in different ways, but that in mathematics, as elsewhere, we especially value the way that is most direct or least wasteful. Our aim is not just to get the job done, but to do it with some sense of style. Introducing this playful approach before introducing derivatives enables students to experience something of the nature of mathematical research, encourages them to try it on their own, and may even get them to think about the value of what they do. Later, many of them will appreciate more deeply the deftness of Newton's method, which duplicates "their" research and does so much more.

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Amortization: An Application of Calculus

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It is very easy to slight the Intermediate Value Theorem and relegate the Monotonicity Theorem to curve-sketching in first-year calculus classes. Here we present a simple application of these theorems to the amortization of a loan. This application is difficult to find in calculus or mathematics of finance texts, but we believe it is well-suited as a small project for first-year calculus students.

Amortization can be viewed as the use of regularly spaced, equal payments for the repayment of a loan to which interest is being added at some fixed rate. Common examples of amortization include home mortgages, car loans, and installment purchases. We will consider only amortizations for which the time between loan payments is equal to the time between interest applications. This would include, for example, a loan being repaid with equal monthly payments while interest is accruing on the loan at some fixed monthly rate. We will use the variables $P =$ original loan principal, $M =$ periodic loan payment, $n =$ number of payments to repay the loan, and $r =$ periodic interest rate. We will assume these quantities are positive, with n being an integer. In the repayment process, part of each payment is applied to the balance of the loan with the remainder applied to the interest. This will be denoted by

$$\begin{aligned} I_m &= \text{interest part of the } m^{th} \text{ payment} = r\cdot(\text{remaining balance}), \\ P_m &= \text{principal part of the } m^{th} \text{ payment} = M - I_m. \end{aligned}$$

Of course, the sum of the principal parts is the original loan principal: $\sum_{i=1}^{n} P_i = P$.

The problem we will consider is to show that fixed valid values of P, M, and n correspond to a unique positive interest rate r. For $1 < m \leq n$,

$$P_m = M - r\left(P - \sum_{i=1}^{m-1} P_i\right).$$

An easy induction argument can then be used to show that for $1 \leq m \leq n$,

$$P_m = (1+r)^{m-1}(M - rP).$$

The equation relating P, M, n, and r can then be found:

$$\begin{aligned} P = \sum_{i=1}^{n} P_i &= \sum_{i=1}^{n} (1+r)^{i-1}(M - rP) \\ &= (M - rP)\cdot\frac{(1+r)^n - 1}{(1+r) - 1} = \frac{M - rP}{r}\cdot\left[(1+r)^n - 1\right]. \end{aligned}$$

Hence $\frac{rP}{M - rP} + 1 = (1+r)^n$, which can be written as

$$\frac{M}{M - rP} = (1+r)^n. \tag{1}$$

It is easy to solve this equation for P, M, and n in terms of the other three variables. However, calculus students will quickly notice the difficulty in attempting to solve it for r. As a straightforward application of the Intermediate Value and Monotonicity Theorems, we will show that for fixed valid values of P, M, and n, there is a unique r satisfying (1) with $0 < r < \frac{M}{P}$.

Before proceeding, we should mention that although (1) is discussed in most mathematics of finance texts which include amortization, their discussions about solving for r in (1) usually consist of writing (1) as $0 = (M - rP)(1+r)^n - M$ and estimating the roots of this polynomial in r using interpolation or a few repetitions of Newton's Method (for example, see [**1**, p. 196] or [**2**, p. 69]). Since it is intuitively clear that a unique positive solution for r exists, it is important that we be able to prove this fact for our mathematical interpretation of amortization.

We begin by letting $f(x) = (M - xP)(1 + x)^n - M$ with $x > 0$. Then $r > 0$ satisfies (1) if and only if r is a zero of f. We will show that f takes on both positive and negative values for $x \in (0, \frac{M}{P}]$. Then, since f is continuous, by the Intermediate Value Theorem we will know that there must be a zero of f in $(0, \frac{M}{P})$.

Notice first that although $r = \frac{M}{P}$ does not make sense in (1), we can compute $f\left(\frac{M}{P}\right) = -M < 0$. Next, since $x > 0$ and n is a positive integer, according to the Binomial Theorem we know that $(1 + x)^n = 1 + xn + \cdots + x^n \geq 1 + xn$. Now for all $x \in (0, \frac{M}{P}]$, we have $M - xP \geq 0$, and hence $(M - xP)(1 + x)^n - M \geq (M - xP)(1 + xn) - M$. Thus by defining $g(x) = (M - xP)(1 + xn) - M$, it follows that $f(r) \geq g(r)$ for all $r \in (0, \frac{M}{P}]$. Furthermore, $nM - P$ is positive, and so $\frac{M}{P} - \frac{1}{n} > 0$. Then for $\hat{r} = \frac{M}{P} - \frac{1}{n} \in \left(0, \frac{M}{P}\right)$, an easy calculation shows $g(\hat{r}) = 0$. (We will leave it to the reader to determine how we knew $\hat{r}$ would work so nicely.) But $f(\hat{r}) \geq g(\hat{r})$, and hence $f(\hat{r}) \geq 0$. Recall that our current goal is to find a value of r in $\left(0, \frac{M}{P}\right)$ for which $f(r) = 0$. So we are done if $f(\hat{r}) = 0$. And if $f(\hat{r}) > 0$, then along with $\frac{M}{P}$ we have found values of r in $(0, \frac{M}{P}]$ for which f takes on both positive and negative values. By the Intermediate Value Theorem, we may conclude that there is a value of r in $\left(0, \frac{M}{P}\right)$ for which $f(r) = 0$. Hence we have proved the existence of $r \in \left(0, \frac{M}{P}\right)$ satisfying (1).

To show the uniqueness of this r, we will use the Monotonicity Theorem to verify that f is monotone decreasing by showing that the derivative

$$\begin{aligned} f'(x) &= (M - xP)\,n(1 + x)^{n-1} - P(1 + x)^n \\ &= (1 + x)^{n-1}[n(M - xP) - P(1 + x)] \end{aligned}$$

is negative for all $x > 0$. Intuitively, $M - rP$ is the part of the first payment applied to the principal. But certainly the principal part of each payment increases with each payment, and so it must be the case that $n(M - rP) \leq P$. Formally, for $1 \leq m \leq n$,

$$\begin{aligned} M - rP &\leq (1 + r)^{m-1}(M - rP), \text{ so} \\ M - rP &\leq P_m, \text{ and} \\ n(M - rP) &\leq \sum_{i=1}^{n} P_i = P. \end{aligned}$$

Also, since r is positive, then $P < P(1 + r)$, and we have $n(M - rP) < P(1 + r)$. Thus

$$(1 + r)^{n-1}[n(M - rP) - P(1 + r)] < 0.$$

This shows that $f'(r) < 0$ for all $r > 0$, and we may conclude that f is monotone decreasing on its domain. Since we have already shown the existence of $r \in \left(0, \frac{M}{P}\right)$ for which $f(r) = 0$, we know that this r is the unique interest rate which corresponds to the fixed values of P, M, and n.

It is no secret that most students struggle to express ideas they understand intuitively using the formal language of mathematics. (It is this difficulty that is probably responsible for the collective groan usually heard when students first encounter word problems in any college mathematics class.) While this application of calculus to amortization is not likely to convince students of the usefulness of mathematics in "real life," it does give them an opportunity to practice translating something they understand intuitively into a problem they can analyze mathematically (with, of course, ample guidance from their instructor). And, at the very least, this problem works well as an elementary exercise in "mathematical modeling" which can be used to illustrate several of the important theorems of first-year calculus. Indeed, the problem generated favorable results when classroom tested in this manner by the authors.

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Reexamining the Catenary

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Some years ago I was called upon to investigate an accident involving the failure of an overhead electrical transmission cable. One of the first tasks was to establish the geometry of the cable's profile, which took the form of a catenary suspended from two supports at different elevations. A search of various textbooks and handbooks of mathematics, mechanics, and engineering practice produced what appeared to be a settled conclusion: when the supports of a catenary are at different elevations, the mathematical complexity precludes a theoretically correct solution, and a parabolic approximation is the recommended approach.

Having since retired, I had the time to revisit this question, convinced that there must be a mathematically correct way to solve the generalized problem. As it turns out, a combination of algebra and a scientific calculator will do the job.

In Figure 1, d, v, l, and g are primary variables; in order to determine one, the values of the other three must be known, since the fixing of any three of them defines the shape and size of the catenary. It is the somewhat elusive parameter, c, however, which governs the relationships between these variables.

If g is the unknown variable, we will find an equation that determines c in terms of d, v, and l.

Although an equation for a catenary may be written in various ways, the simplest form for algebraic purposes is:

$$y = c\cosh\frac{x}{c}, \tag{1}$$

the derivation of which may be found in many texts. An equation for s is readily found by applying the familiar formula for arc length:

$$s = \int_0^x \sqrt{1 + (\sinh x/c)^2}\, dx = \int_0^x \cosh(x/c)\, dx = c\sinh\frac{x}{c}. \tag{2}$$