Corollary. *The multiplicative group $\mathbb{C}^*$ is isomorphic to $\mathbb{S}^1$.*

Proof. Using the maps $z \mapsto e^{2\pi i z} (z \in \mathbb{C})$ and $r \mapsto e^{2\pi i r} (r \in \mathbb{R})$, it is clear that $\mathbb{C}^* \cong \mathbb{C}/\mathbb{Z}$ and $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$. On the other hand the map f in the proof above maps $\mathbb{Q}$ onto $\mathbb{Q}$, and hence $\mathbb{Z}$ onto $\mathbb{Z}$. This induces an isomorphism $\mathbb{C}/\mathbb{Z} \cong \mathbb{R}/\mathbb{Z}$, as required.

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Tangents without Calculus

Jorge Aarão (jaarao@benson.mckenna.edu),
Claremont McKenna College, Claremont, CA 91711

In pre-calculus courses we often teach our students about polynomial division, and use the division algorithm in factoring polynomials. I would like to suggest another interesting application of polynomial division.

Here's the no-calculus rule for finding tangent lines to polynomials.

> The line $y = mx + b$ is tangent to the graph of the polynomial $p(x)$ at $x = a$ if and only if $mx + b$ is the remainder of the quotient $p(x)/(x-a)^2$.

For example, since

$$x^2 - 2x^2 + x + 1 = (x+2)(x-2)^2 + (5x-7),$$

$y = 5x - 7$ is tangent to $y = x^3 - 2x^2 + x + 1$ at $x = 2$.

While this rule may not be as simple as the calculus method for finding tangent lines, from a pre-calculus point of view it is not only elementary but also has a very intuitive, geometric justification.

First let's answer the question: when is the x-axis tangent to the graph of a polynomial? Let $f(x)$ be a polynomial and let $x = a$ be a root of f, then, as suggested by Figure 1, the x-axis is tangent to the graph of f exactly when $x = a$ is (at least) a double root of f. Equivalently, $(x-a)^2$ divides $f(x)$ without remainder.

The question of whether a line $y = mx + b$ is tangent to the graph of a polynomial $p(x)$ at $x = a$ can be reduced to the previous case by setting $f(x) = p(x) - (mx + b)$. Now $y = mx + b$ is tangent to $y = p(x)$ at $x = a$

$\Leftrightarrow$ the x-axis is tangent to $y = f(x)$ at $x = a$
$\Leftrightarrow (x-a)^2$ divides $f(x)$ without remainder
$\Leftrightarrow mx + b$ is the remainder of the quotient $p(x)/(x-a)^2$.

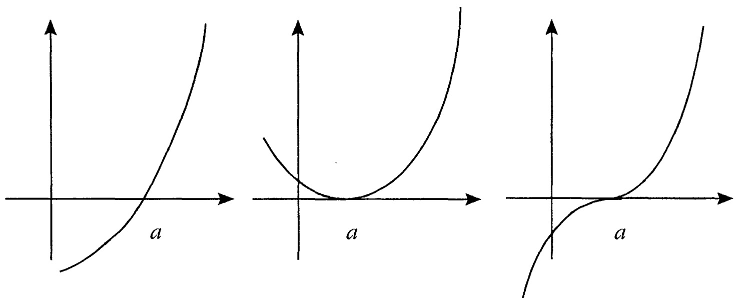

Figure 1

In my opinion there are various arguments favoring the presentation of this example to a pre-calculus class. First, it is a non-trivial application of polynomial division. Second, the early exposure to the concept of tangent lines will surely facilitate the transition to the calculus approach. Finally, for those classes with access to graphing devices it could be profitable to perform the division and then compare the graphs of $y = p(x)$ and $y = mx + b$.

I have found that students get a better appreciation of the power of calculus when they (later) compare the easier-to-use calculus method to the above argument. So, in a standard calculus course, the rule can be presented as an interesting footnote or even as an exercise, using calculus to prove that the tangent line is obtained from the remainder.

Finally, the rule can even make a guest appearance in the second calculus course, when students learn about Taylor series or second-order approximation. For functions $f(x)$ which have a Taylor expansion around $x = a$ we can write

$$f(x) = f(a) + f'(a)(x-a) + (x-a)^2 g(x),$$

and in the special case where f is a polynomial then g is also a polynomial, and we recover our rule.

Erratum

Professor Kathy Buxie, of Ohio University—Lancaster (buxie@oak.cats.ohiou.edu) notes that in the March *Journal* there appears on page 119 the startling equation

$$\frac{2}{1-6} = -1.$$

While this is correct (mod 3), the numerator of the fraction should be 5.