$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{11}{3} & 0 \\ 0 & 0 & \frac{10}{11} \end{bmatrix}$$

Letting $B = B_1^T B_2^T B_3^T = \begin{bmatrix} 1 & -2/3 & 7/11 \\ 0 & 1 & -5/11 \\ 0 & 0 & 1 \end{bmatrix}$, we have $B^{-1} = \begin{bmatrix} 1 & 2/3 & -1/3 \\ 0 & 1 & 5/11 \\ 0 & 0 & 1 \end{bmatrix}$.

For a square root of the diagonal matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 11/3 & 0 \\ 0 & 0 & 10/11 \end{bmatrix}$, let $C = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{33}/3 & 0 \\ 0 & 0 & \sqrt{110}/11 \end{bmatrix}$.

Then $C^{-1} = \begin{bmatrix} \sqrt{3}/3 & 0 & 0 \\ 0 & \sqrt{33}/11 & 0 \\ 0 & 0 & \sqrt{110}/10 \end{bmatrix}$, and we obtain

$$Q = ABC^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{3} & \frac{-2\sqrt{33}}{33} & \frac{-2\sqrt{110}}{55} \\ \frac{\sqrt{3}}{3} & \frac{4\sqrt{33}}{33} & \frac{-3\sqrt{110}}{110} \\ \frac{-\sqrt{3}}{3} & \frac{2\sqrt{33}}{33} & \frac{-7\sqrt{110}}{110} \\ 0 & \frac{\sqrt{33}}{11} & \frac{3\sqrt{110}}{55} \end{bmatrix}$$

and

$$R = CB^{-1} = \begin{bmatrix} \sqrt{3} & \frac{2\sqrt{3}}{3} & \frac{-\sqrt{3}}{3} \\ 0 & \frac{\sqrt{33}}{3} & \frac{5\sqrt{33}}{33} \\ 0 & 0 & \frac{\sqrt{110}}{11} \end{bmatrix}.$$

Although the foregoing method of orthonormalizing a matrix is not frequently used, as compared to the standard Gram-Schmidt process, its procedure is simple, and it is able to avoid the inaccuracy problems inherent in the latter method.

This note can be viewed as an explication of the idea found in section 4 of [**1**], in which many references contain detailed information on the decomposition of real and complex matrices.

Acknowledgment. The author would like to thank Professors Thomas A. Farmer and the Editor for suggesting the title of this note and improving its content.

Reference

1. Roger A. Horn and Ingram Olkin, When does $A * A = B * B$ and why does one want to know, *American Mathematical Monthly* **103** (1996) 470–482.

The n^{th} Derivative Test and Taylor Polynomials Crossing Graphs

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In [**2**], Samuel B. Johnson develops a nice criterion for determining when the graphs of Taylor polynomials and their associated functions will cross.

Theorem 1 (Johnson). *Suppose a real-valued function f can be represented by its Taylor Series on a neighborhood U of a point a, and that there is an mth degree polynomial p for which $p^{(k)}(a) = f^{(k)}(a)$ for $k = 0, \ldots, m$. Suppose further that $f^{(n)}(a)$ is the first nonzero derivative for some $n > m$. Then p's graph crosses f's graph at a if and only if n is odd.*

I would like to point out that the nth Derivative Test offers a quick proof of his criterion, and offers additional information on whether the graph of a Taylor polynomial crosses, stays above (locally of course), or stays below the graph of its associated function.

Theorem 2 ([3, p. 216]). *Suppose a real-valued function g is defined on a neighborhood U of a point a, where $g^{(k)}(a) = 0$ for $k = 1, \ldots, n-1$ and $g^{(n)}(a) \neq 0$, where $n \geq 2$. Assume that each such $g^{(k)}$ is continuous on U. Then the following must hold:*

(a) If n is even and $g^{(n)}(a) < 0$, then g has a local maximum at a;
(b) If n is even and $g^{(n)}(a) > 0$, then g has a local minimum at a;
(c) If n is odd, then g has neither a local maximum nor a local maximum at a.

The reader will recognize Theorem 2, the so-called "nth Derivative Test," as a generalization of the Second Derivative Test from Calculus I. This result is often proved in Introductory Analysis by applying Taylor's Theorem (the continuity of $g^{(n)}(x)$ on U assures us that the sign of $g^{(n)}(a)(x-a)^n/n!$ will govern the sign of $g(x) - g(a)$ on a neighborhood of a). No infinite series need be considered.

The following result modestly generalizes Johnson's Theorem, as it tells us about the behavior of the graphs of a Taylor polynomial and its associated function when n is even.

Theorem 3. *Suppose a real-valued function f is defined on a neighborhood U of a point a and suppose that there is an mth degree polynomial p having $p^{(k)}(a) = f^{(k)}(a)$ for $k = 0, \ldots, m$. Suppose further that $f^{(n)}(a)$ is the first nonzero derivative for some $n > m$, and that each $f^{(k)}$ is continuous on U for $k = 1, \ldots, n$. Then the following must hold:*

(a) If n is even and $f^{(n)}(a) < 0$, then f's graph lies below p's graph on a neighborhood of a;
(b) If n is even and $f^{(n)}(a) > 0$, then f's graph lies above p's graph on a neighborhood of a;
(c) If n is odd, then p's graph crosses f's graph at a.

Proof. Define $g(x) = f(x) - p(x)$. Then $g^{(k)}(a) = 0$ for $k = 1, \ldots, n-1$ and $g^{(n)}(a) \neq 0$, so the conclusions of Theorem 2 apply to $g(x)$. In case (a), the function g has a local maximum at a. Since $g(a) = 0$, we see that $g(x) \leq 0$, which implies that $f(x) \leq p(x)$. Therefore f's graph lies below g's graph. The proofs of (b) and (c) are similar. ∎

It is common in the initial stages of Calculus I to investigate limits via graphs, using technology where appropriate. A nice example is

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

considered in [**1**]. An initial viewing suggests that the graph of $f(x) = \frac{\tan x - x}{x^3}$ is well-behaved near 0, and that the limit in question is $1/3$ (Figure 1). A complacent student might stop at this point, and announce that the limit is $1/3$, whereas more industrious students might zoom in more about the point $(0, 1/3)$. With enough zooming, the TI-83, Maple, and Mathematica all give very fractured pictures of what is going on (Figure 2), possibly confusing the students. At this point, the instructor could take the opportunity to do a detailed numerical analysis, jump ahead to L'Hôpital's Rule, or possibly deliver a lecture on the dangers and frailties of technology. Let's leave this Calculus I teaching scenario alone and consider the problem by using Theorem 3.

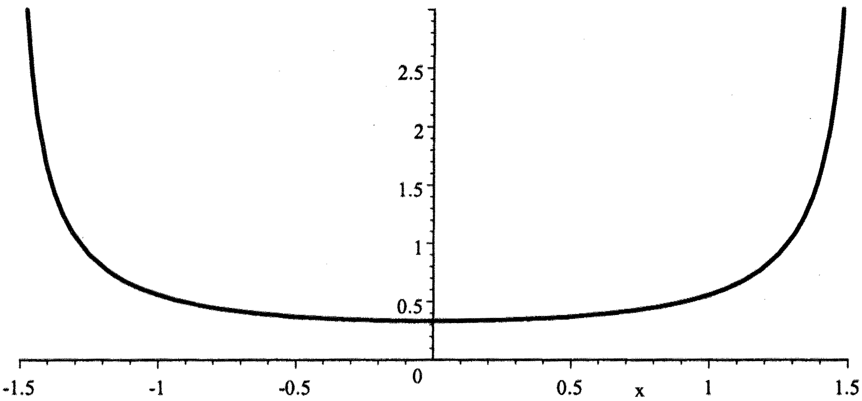


Figure 1. An initial view of the graph of $\frac{\tan x - x}{x^3}$

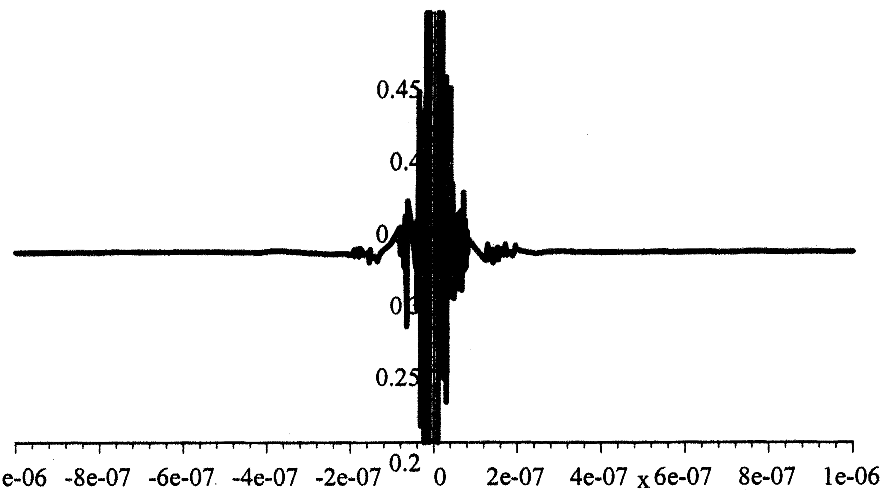


Figure 2. A closer view of the "graph" of $y = \frac{\tan x - x}{x^3}$

Let $g(0) = 1/3$ and $g(x) = f(x)$ for $x \neq 0$, so the Taylor series of $g(x)$ is

$$g(x) = \frac{1}{3} + \frac{2}{15}x^2 + \frac{17}{315}x^4 + O(x^6).$$

For the Taylor polynomial $p(x) = \frac{1}{3} + \frac{2}{15}x^2$, observe that $m = 2$ and $n = 4$ in Theorem 3, and since $g^{(4)}(0) = 4! \cdot 17/315 > 0$, we can be assured that g's graph stays above the parabola $p(x) = \frac{1}{3} + \frac{2}{15}x^2$ on a neighborhood of 0. This provides us strong evidence that the initial picture was reasonable and that the zoom-in view in Figure 2 is garbage, at least below the parabola $p(x)$.

Taylor's Theorem offers an insightful proof of Theorem 2 and, through Theorem 3, some insight into the geometry of function graphs. A strong calculus class might enjoy heuristic arguments for Theorems 2 and 3, and their ramifications for technology and limit problems such as the one discussed above.

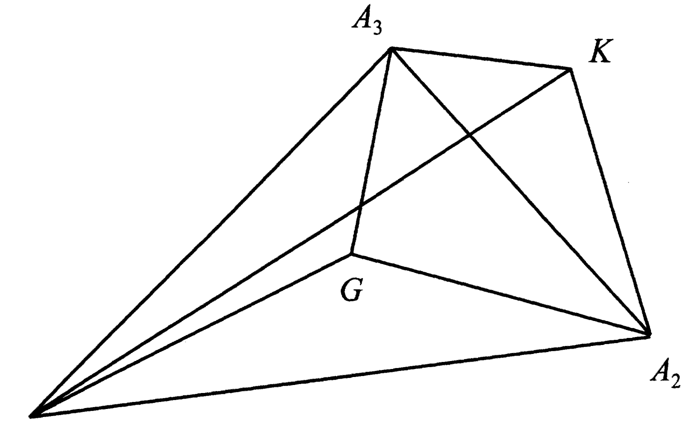
References

1. William Emerson and Lou Talman, *Mathematica Notebooks for Calculus*, Revised Edition, Auraria Reprographics, Denver, August, 2000.
2. Samuel B. Johnson, When do approximating polynomials cross graphs of approximated functions, *The College Mathematics Journal* **31** (2001) #1 57–58.
3. Lay, Steven R., *Analysis with an Introduction to Proof*, 2nd ed., Prentice-Hall, 1990.

Mathematics Without Words

Norman Schaumberger (nschaumber@aol.com) writes, "A familiar textbook exercise in plane coordinate geometry is to show that the sum of the vectors from the centroid of a triangle to the three vertices is zero. A less familiar result is that if the sum of the squares of the distances from a point in space to the three vertices of a triangle is a minimum, then the point is the centroid of the triangle."

Here is why.



$$\sum_{i=1}^{3} \overrightarrow{GA_i} = \vec{0} \Rightarrow \sum_{i=1}^{3} |KA_i|^2 = \sum_{i=1}^{3} \overrightarrow{KA_i} \cdot \overrightarrow{KA_i} = \sum_{i=1}^{3} \left(\overrightarrow{KG} + \overrightarrow{GA_i}\right) \cdot \left(\overrightarrow{KG} + \overrightarrow{GA_i}\right)$$

$$= 3\overrightarrow{KG} \cdot \overrightarrow{KG} + 2\overrightarrow{KG} \cdot \sum_{i=1}^{3} \overrightarrow{GA_i} + \sum_{i=1}^{3} \overrightarrow{GA_i} \cdot \overrightarrow{GA_i} = 3|KG|^2 + \sum_{i=1}^{3} |GA_i|^2.$$

