Incidentally, in the end, I sent four copies of the letter to my sister and her husband, perhaps breaking the rules in spirit but not exactly breaking the chain. If she had sent 16 copies back to me we would have had quite a bit of exponential fun!

References

1. J. O. Case, The chain letter: an example of exponential growth, *Mathematics Teacher* **80** (1987), 114–115.
2. S. M. Ross, *Introduction to Probability Models*, 7th ed., Academic Press, San Diego, 2000.
3. D. J. Thuente, Chain letters: a poor investment unless..., *Two-Year College Mathematics Journal* **13** (1982), 28–35.

The Distance Between Two Graphs

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Given the graphs of two functions—graphs that do not intersect—one might wonder about the minimum distance between them and how it is calculated. On first consideration, it would seem that a solution to this problem requires multivariable calculus because the distance between points on the two graphs is a function of their x-coordinates. But here is a way to solve some such problems in a beginning calculus course using single variable methods.

As an example, let us examine the graphs of $f(x) = e^x$ and $g(t) = -t^2$ (see the figure). It appears that the minimum distance between these graphs occurs along a line segment that is perpendicular to both graphs. If this is indeed the case, then we can find the values of x and t for which the line segment joining (x, e^x) and $(t, -t^2)$ is perpendicular to the tangent lines to the graphs at these two points. This idea is represented in the system

$$\frac{-1}{e^x} = \frac{e^x - (-t^2)}{x - t} = \frac{-1}{-2t}.$$

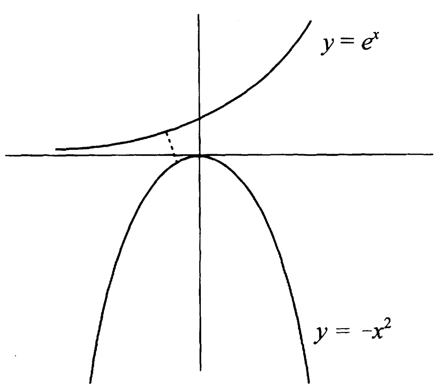

Figure 1.

A computer algebra system or graphing calculator gives us approximate values for x and t leading to the points $(-0.608, 0.544)$ on the graph of f and $(-0.272, -0.074)$ on the graph of g. The distance between these points is 0.578.

To prove that our procedure is valid under the stated conditions, suppose we are given differentiable functions f and g for which the set $\{|PQ| : P = (x, f(x)), Q = (t, g(t))\}$ attains a minimum value $d = |P_0Q_0|$, where neither P_0 nor Q_0 is an endpoint of one of the graphs. We want to prove that the line segment P_0Q_0 is perpendicular to the tangent lines to f's graph at P_0 and to g's graph at Q_0. To do this, we hold the endpoint $Q_0 = (t_0, g(t_0))$ fixed and let $P = (x, f(x))$ vary. The square of the distance from Q_0 to P is the differentiable function $D(x) = (x - t_0)^2 + (f(x) - g(t_0))^2$ whose minimum value occurs where $D'(x) = 0$:

$$2(x - t_0) + 2(f(x) - g(t_0)f'(x) = 0.$$

So, $D(x)$ is minimum when $f'(x) = 0$ and $x = t_0$, or when $f'(x) \neq 0$ and

$$\frac{f(x) - g(t_0)}{x - t_0} = \frac{-1}{f'(x)}.$$

In either case, the segment P_0Q_0 of minimum length is perpendicular to f's graph at the point P_0. By reversing the roles of f and g, we get the other part of the claim and this completes the proof.

Of course, in practice, finding the lengths of line segments PQ that are perpendicular to the tangent lines at P and Q gives us only candidates for the distance between the graphs. We would need to determine whether the length of one them actually is the minimum distance between the two graphs [cf. c) below].

Suggested Problems

a) Find the minimum distance between the graphs of $f(x) = e^x$ and $g(t) = \ln t$.

b) Find the minimum distance between the graphs of $f(x) = 1 + (x + 1)^2$ and $g(t) = 1 - 1/t$ $(t > 0)$.

c) $f(x) = x + 2 + \sin x$ and $g(t) = t$.

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The Alternating Harmonic Series

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The following derivation that the alternating harmonic series converges to $\ln 2$ is more elementary than the standard one in the textbooks or the several that have appeared in journals (e.g., [**1**], [**2**], [**3**], [**4**], [**5**]) in that it does not use integrals or infinite series (except trivially).

Define

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n},$$