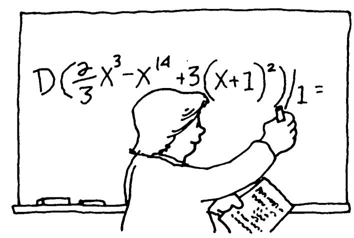
CLASSROOM CAPSULES



A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Frank Flanigan.

Clapping Music—a Combinatorial Problem

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A composition entitled "Clapping Music" by the well-known American musician Steve Reich provides the basis for a series of mathematical problems in combinatorics and group theory.

"Clapping Music" is a piece for two performers, each of whom repeatedly claps a pattern determined by the composer. The pattern of twelve beats consists of a steady sequence of three claps, followed by a pause, followed by two claps, a pause, one clap, a pause, two claps, and a final pause; see Figure 1. An eighth note signals a clap, an eighth rest a pause. As the musical notation indicates, each pause and each clap are of equal duration.

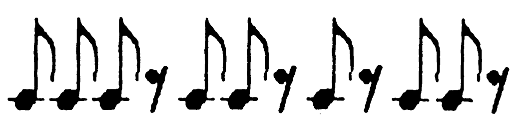

Figure 1

One performer will repeat this pattern throughout the piece. The second performer, after repeating the pattern a number of times simultaneously with the first performer, will abruptly move one beat ahead. Regarding the pattern cyclically, the pattern of the second performer is still the same as before, but shifted in time. Against the pattern of the first performer, the listener will hear two claps, a pause, two claps, a pause, one clap, a pause, two claps, a pause, and one clap; see Figure 2.

Figure 2

After a number of repetitions of the pattern at one beat ahead of the first performer, the second performer will then move two beats ahead, and so on, until the second performer has cycled through all twelve possible displacements of the pattern. The piece concludes with the two performers again clapping the pattern in unison. (A recording of the piece, performed by Reich and another musician, is most readily available on the album, *Steve Reich*: *Early Works*, Elektra/Asylum/Nonesuch Records 9 79169-4, 1987. The composer discusses the piece on pages 64–66 of "Notes on Compositions, 1965–1973," in his *Writings about Music*, The Press of the Nova Scotia College of Art and Design, Halifax, 1974.)

Music is a means of communication. As such, it is reasonable to ask about the information it carries. One aspect of this is the number of choices possible to a composer in creating his composition. If we first limit our attention to arrangements of 8 claps and 4 pauses within a 12-beat measure (the raw material of Reich's pattern), we find, of course, $\binom{12}{8} = 495$ possibilities. In a composition such as this, however, it is reasonable to impose some restrictions on the patterns we will allow; not all of the 495 possible patterns are of equal esthetic or intellectual interest. For example, Reich chooses to begin the piece with a clap; this serves to let us know the piece has begun! Also, Reich has not allowed consecutive pauses. With these two restrictions, we can now count the arrangements by considering the placement of 4 clap-pause combinations and 4 claps, arriving at $\binom{8}{4} = 70$ possibilities.

The structure of "Clapping Music" leads us to impose an equivalence relation on these 70 patterns. If we regard "Clapping Music" as a piece without beginning that could cycle endlessly, we would regard as equivalent any two patterns that produced the same piece. In other words, two patterns should be equivalent if they are cyclic permutations of each other. Thus to count the number of pieces that could be made by cyclically permuting a pattern of 8 claps and 4 pauses, with no consecutive pauses, we can apply Burnside's lemma [see, for example, Alan Tucker, *Applied Combinatorics*, second edition, John Wiley, New York, 1984, p. 343]:

> *Let G be a finite permutation group of order n on a set S of patterns. For each $\varphi \in G$, let $n(\varphi)$ be the number of patterns in S fixed by φ. Then the number N of inequivalent patterns is*

$$N = \frac{1}{n} \sum_G n(\varphi).$$

The set in our application consists of the twelve-beat patterns; the group suggested by the musical composition is the cyclic group generated by the twelve-cycle

$$\sigma = (1\ 12\ 11\ 10\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2).$$

In order to create a collection of patterns that is closed under the action of the group, we must retreat a bit and allow patterns that begin with a pause. Then, for example, the identity σ^0 fixes all 105 patterns: the 70 that begin with a clap and the $\binom{7}{3} = 35$ that begin with a pause and end with a clap (place 3 clap-pause pairs and 4 claps after the initial pause). There are no patterns fixed by σ^i if $i = 1, 2, 4, 5, 7, 8, 10$, or 11, because none of these permutations permits orbits containing only pauses. The permutations σ^3 and σ^9 fix 3 patterns each, determined by the position of the pause within an initial grouping of 2 claps and 1 pause. Finally, σ^6

fixes $\binom{4}{2} = 6$ patterns that begin with a clap (place 2 clap-pause combinations and 2 claps) and $\binom{3}{1} = 3$ patterns that begin with a pause (place 1 clap-pause combination and 2 claps after the initial pause and before the final clap). Applying Burnside's lemma gives

$$\tfrac{1}{12}(105 + 0 + 0 + 3 + 0 + 0 + 9 + 0 + 0 + 3 + 0 + 0) = 10$$

different pieces.

Another formulation of the solution to this problem that does not require the consideration of patterns that begin with a rest is obtained by allowing an 8-cycle τ to act on the 70 patterns of 4 clap-pause combinations and 4 claps. Now only τ^0, τ^2, τ^4, and τ^6 fix nonzero numbers of patterns, namely 70, 2, 6, and 2, respectively. Applying Burnside's lemma gives

$$\tfrac{1}{8}(70 + 0 + 2 + 0 + 6 + 0 + 2 + 0) = 10$$

different pieces, as before.

What are these 10 pieces? Now that their number has been found to be small, it becomes a reasonable task to find them all. Because of the equivalence relation, we may choose as a representative of each piece a pattern that begins with a maximal number of consecutive claps and hence ends with a pause. We can then encode each pattern as four positive integers that sum to 8. Thus, for example, $3, 2, 1, 2$ gives Reich's original pattern of three claps, a pause, two claps, a pause, one clap, a pause, two claps, and a pause. The 10 pieces result from the following 10 representative patterns:

$5, 1, 1, 1$	$3, 1, 3, 1$
$4, 2, 1, 1$	$3, 2, 2, 1$
$4, 1, 2, 1$	$3, 2, 1, 2$
$4, 1, 1, 2$	$3, 1, 2, 2$
$3, 3, 1, 1$	$2, 2, 2, 2.$

A consideration of this list shows that there is one additional criterion that Reich would require of a pattern, namely, that cyclically permuting the pattern gives twelve distinct patterns. The patterns $3, 1, 3, 1$ and $2, 2, 2, 2$ would return after only six and three applications of σ, respectively. Thus there are only 8 inequivalent pieces that allow cycling through 12 distinct patterns.

Since it was not difficult to list the 10 inequivalent patterns, why bother with Burnside's lemma in this situation? One reason is the excuse provided for possible generalizations. Suppose that there are $2n$ claps and n pauses, where n is relatively prime to 3. If σ now denotes the natural $3n$-cycle, then σ^d fixes a nonzero number of patterns only if the order of σ^d divides n. A similar result holds if τ denotes a $2n$-cycle acting on n clap-pause combinations and n claps. Can the students find any interesting combinatorial identities here?

Finally, what is special or unique about Reich's $3, 2, 1, 2$ pattern among the representatives of the eight possible pieces? It is one of only two ($4, 1, 2, 1$ being the other) that has no consecutive repetitions of the number of claps between two consecutive pauses.

After hearing a recording of this music, students enjoy analyzing the piece, finding that the esthetic appeal and intellectual attractiveness of "Clapping Music" can, in part, by explained by three factors: the complexity of pattern allowed by 12 beats composed of four pauses and eight claps, the variations that result from the application of such a simple cyclic permutation, and the syncopation provided by the particular 3, 2, 1, 2 pattern used by Reich. Students find these considerations far more exciting than counting beaded necklaces! The use of materials from the humanities in mathematics classrooms can be invaluable in maintaining students' interest.

'Hidden' Boundaries in Constrained Max-Min Problems

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In my first period Calc III class we considered the problem of finding the minimum distance from the origin to the paraboloid $z = 4 - x^2 - 4y^2$. The first octant of the constraint surface is shown in Figure 1.

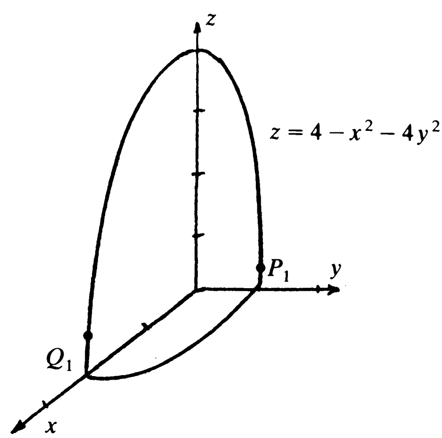

Figure 1

Minimizing distance-squared and replacing x^2 by $4 - 4y^2 - z$ gives the following function of two variables to be minimized

$$D(y, z) = 4 - 4y^2 - z + y^2 + z^2 = 4 - 3y^2 + z^2 - z.$$

Setting the partial derivatives D_y and D_z to zero gives

$$D_y = -6y \quad = 0$$
$$D_z = 2z - 1 = 0.$$

Thus the only critical point of D is $y = 0$, $z = 1/2$. Solving for x on the paraboloid gives the points $Q_1(\sqrt{7/2}, 0, \frac{1}{2})$ and $Q_2(-\sqrt{7/2}, 0, \frac{1}{2})$. Q_1 is shown in Figure 1.

It was clear to the students that the distance from the origin to Q_1 or Q_2 is not minimal since the point $(0, 1, 0)$ is closer. I was at a loss to explain why the critical points did not include the expected global minimum, but I was saved when the first period ended.