COMPUTER CORNER

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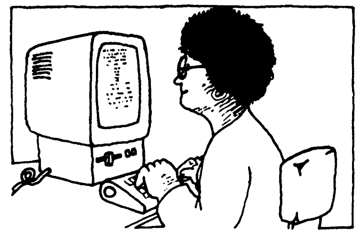

In this column, readers are encouraged to share their expertise and experiences with computer technology as it relates to college mathematics. Articles illustrating how computers can enhance pedagogy, solve mathematics problems, and model real-life situations are especially welcome.

Classroom Computer Capsules feature new examples of using the computer to enhance teaching. These short articles demonstrate the use of readily available computing resources to present or elucidate familiar topics in ways that can have an immediate and beneficial effect in the classroom.

Send submissions to the editor-elect, Underwood Dudley (see inside cover for address).

CLASSROOM COMPUTER CAPSULES

Clock Hands Pictures for 2 x 2 Real Matrices

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Inexperienced students may legitimately wonder, "What, *really*, is an eigenvalue?" This is a subtle question, and unfortunately it calls for a variety of ideas typically reserved for later in the curriculum. However, one answer that students can immediately grasp consists of a simple demonstration via technology that converts the algebraic meaning (coincidence of Ax and a multiple of $x \neq 0$) to a geometric picture. With the "clock hands" graphics program described below, technology serves to communicate mathematical ideas in a lovely way that captures students' attention.

We first saw this idea in the linear algebra curricular sessions (of which the first author was an organizer) at the January 1994 Joint Mathematics Meetings, where Gil Strang also learned of it. He has subsequently mentioned it during pedagogical talks, one of which (DePauw University, March 19, 1994) motivated a *CMJ* note describing a variant of the basic idea (where A is replaced by $A + I$) [**3**].

Recall that the nonzero vector $x \in \mathbb{R}^n$ is an eigenvector of an $n \times n$ matrix A if $Ax = \lambda x$ for some scalar λ. In $\mathbb{R}^2$, therefore, if a 2×2 real matrix A has real eigenvalues, then the vectors $x \in \mathbb{R}^2$, $x \neq 0$, and Ax will fall on the same line through the origin when plotted in $\mathbb{R}^2$. This concept is incorporated into the clock hands graphics program, which is rich both in elementary concepts that the student can discover and in questions for mathematicians of all levels to explore.

Sweeping hands. The clock hands graphics program (for *Maple* and *Mathematica* code, see http://www.math.macalester.edu/~kroschel) uses the fact that the unit vectors $x(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ pass through every normalized vector of $\mathbb{R}^2$ as θ runs from 0 to 2π. For a given 2×2 matrix A, the vector $x(\theta)$ is plotted simultaneously with the vector $Ax(\theta)$. Then θ is allowed to vary from 0 to 2π in discrete increments, so we get the plots of $x(\theta)$ and $Ax(\theta)$ for each value of θ. When this is done with dynamic graphics, the user can see the vector $x(\theta)$ sweep out the unit circle as $Ax(\theta)$ sweeps out an ellipse, as in Figure 1.

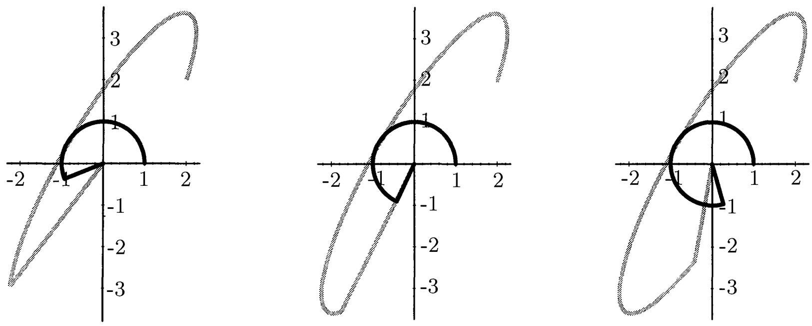

Figure 1. The clock hands for $A = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}$, for three values of θ.

That the image of the unit circle under A is an ellipse (possibly degenerate) can be seen by looking at the singular value decomposition of A ([**2**]). (This parallels the observation in [**1**] that the eigenpictures described in [**3**] also form ellipses.)

If the 2×2 real matrix A has real eigenvalues, then for some θ the vectors $x(\theta)$ and $Ax(\theta)$ are collinear (pointing in the same direction or opposite directions depending on whether the eigenvalue is positive or negative). The vector $x(\theta)$ is then an eigenvector, and the absolute value of the eigenvalue is the length of $Ax(\theta)$. Note that both $x(\theta)$ and $-x(\theta)$ are eigenvectors, so that there are four points at which $x(\theta)$ and $Ax(\theta)$ will line up. For example, the central picture in Figure 1 shows an eigenvalue–eigenvector pair for the matrix $A = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}$.

Observations and examples. By entering a variety of 2×2 matrices, we begin to see patterns that lead to some interesting questions. For example:

- The vector sweeping out the ellipse appears to move at varying speeds: faster near the minor axis of the ellipse and slower near the major axis. Why?

The vector $Ax(\theta)$ sweeps out area at a constant rate! The unit vector $x(\theta)$ turns at a constant rate, sweeping out a circular sector of area $\Delta\theta$ as the angle θ increases by an amount $\Delta\theta$, and $Ax(\theta)$ simultaneously sweeps out an area $(\det A)\Delta\theta$ as it moves from $Ax(\theta)$ to $Ax(\theta + \Delta\theta)$. The vector $Ax(\theta)$ does not have to move very much when its length is near its maximum in order to sweep out this area, but $Ax(\theta)$ may have to move quite a bit to maintain this area when it is near its minimum length. This accounts for the visible difference in speed.

- For some matrices, the vectors $x(\theta)$ and $Ax(\theta)$ both move counterclockwise; for other matrices, $Ax(\theta)$ moves clockwise, opposite to $x(\theta)$. Why does this happen?

This phenomenon too is due to $Ax(\theta)$ sweeping out a constant area—but an area that is signed. Its sign is determined by the sign of $\det(A)$. So $Ax(\theta)$ will move in a counterclockwise direction when the two eigenvalues of A have the same sign and will move clockwise when the eigenvalues are opposite in sign.

- Try a Jordan block matrix, $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$. What happens?

Note that for $A = \lambda I$, the vectors $x(\theta)$ and $Ax(\theta)$ are always collinear since any nonzero vector in $\mathbb{R}^2$ is an eigenvector. The Jordan block, however, has only one "eigen-axis", and the vectors $x(\theta)$ and $Ax(\theta)$ kiss when x is an eigenvector and then separate again, but never actually cross. After you show the graphics of this example to the students, ask them to describe what they observe and to explain the cause.

- Suppose a matrix A has complex eigenvalues. What will happen?

In this case, the vectors $x(\theta)$ and $Ax(\theta)$ never line up, but there is a point of nearest (and farthest) passage. What does the vector $x(\theta)$ represent at such a point? Does the angle between $Ax(\theta)$ and $x(\theta)$ at this point have any meaning? These are good questions for further exploration; we do not yet know their answers.

When singular values and singular vectors [**2**] have been discussed in class, their geometric implications can be illustrated with the clock hands graphics program. The lengths of the major and minor axes of the ellipse formed by $Ax(\theta)$ correspond to the maximum and minimum singular values of A, respectively. In addition, the unit vectors $x(\theta)$ at which the major and minor axes occur are the corresponding (right) singular vectors.

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References

1. H. Fraser and P. Zizler, Eigenpictures and singular vectors of a matrix, *College Mathematics Journal* 28 (1997) 1–4.
2. R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
3. S. Schonefeld, Eigenpictures: Picturing the eigenvector problem, *College Mathematics Journal* 26 (1995) 316–319.

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Geometric Characterization of the Shortest Path in a Tetrahedron

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The problem of finding the closed path of minimum length that touches all four faces of a regular tetrahedron [**2**] was posed by Professor Igor Fedorovich Sharygin on the 1993 Moscow Mathematical Olympiad, a contest for high school students. As I was the only student to solve the problem, it remains one of my favorites. Your readers may enjoy the geometric solution below, which is a combination of Professor Sharygin's solution and mine.