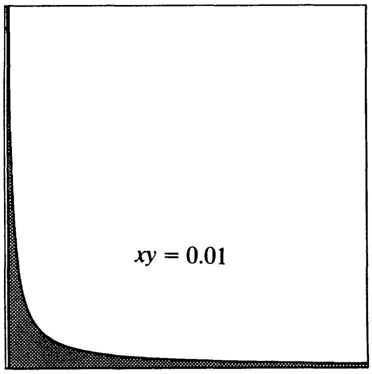

Figure 4
area = $-0.01 \ln(0.01)$.

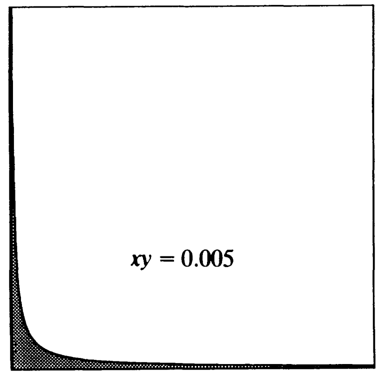

Figure 5
area = $-0.005 \ln(0.005)$.

Summing Geometric Series by Holding a Tournament

Vincent P. Schielack, Jr., Texas A&M University, College Station, TX 77843

The purpose of this note is the presentation of an alternate interpretation of geometric series, casting them in terms of their relation to single-elimination tournaments. Aside from the obvious classroom motivational advantages which result, this interpretation allows geometric series to be summed without resorting to the device of telescoping sums, which appears somewhat prestidigitatorial to many students.

We begin with a simple observation: In a regular single-elimination tournament of m teams, $m-1$ games are required to determine the champion. This is due to the fact that each game produces one loser, and all but one team must lose. In particular, if 2^n teams are entered, we require $2^n - 1$ games. However, we could also calculate directly the total number of games by counting the number of games played during each round of the tournament and summing. Thus, the total number of games played in the tournament is $1 + 2 + 2^2 + \cdots + 2^{n-1}$ (listed in reverse order, from finals to first round). Equating these two expressions for the total number of games in the tournament yields the sum of a geometric series,

$$1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

This particular result is known to practically everyone who has ever followed a standard single-elimination tournament. However, we can generalize this result by modifying the number of teams in each game.

Consider a game in which t teams are matched simultaneously with the outcome of one winner and $t-1$ losers. (Game shows such as *Wheel of Fortune* and *Jeopardy* pit three competitors.) Now suppose that we hold a tournament of these t-team games, beginning with t^n teams, where n is a positive integer. As in the previous example, we can count the total number of games necessary to determine the champion either directly or indirectly. Counting directly, there are t^{n-1} first-round games, t^{n-2} second-round games,..., t semifinal games, and a single "finals" game. Thus, the total number of games played in the tournament is $1 + t + t^2 + \cdots + t^{n-1}$. Counting indirectly, we need only divide the total number of losers, $t^n - 1$, by the number of losers per game, $t - 1$, to obtain $(t^n - 1)/(t - 1)$

games. Equating these expressions for the total number of games yields

$$1+t+t^2+\cdots+t^{n-1}=\frac{t^n-1}{t-1}. \tag{1}$$

Although we derived (1) by considering only integral $t>1$ and positive integral n, it forms an algebraic identity valid for all complex $t \neq 1$. Replacing n with $n+1$ gives the more standard form

$$1+t+t^2+\cdots+t^{n}=\frac{t^{n+1}-1}{t-1},$$

valid for all whole numbers n.

Weighted Means of Order r and Related Inequalities: An Elementary Approach

François Dubeau, Collège militaire royal de Saint-Jean, Saint-Jean-sur-Richelieu, Québec, Canada, J0J 1R0

The aim of this note is to present the properties of weighted means of order r using elementary techniques of analysis. In particular the increasing property of the weighted mean of order r, as a function of r, is proved using a more elementary technique than the standard proof.

Let r be any nonzero real number and let us consider the function $\phi(x)=x^r$ defined on the interval $I=(0,+\infty)$. Using the mean value theorem, for any fixed strictly positive real number a, we have

$$x^r=a^r+r\xi_a^{r-1}(x-a) \tag{1}$$

where ξ_a is between x and a. It follows that $\phi(x)=x^r$ is strictly increasing (decreasing) if $r>0$ ($r<0$).

Using the Taylor expansion of order 1, we have

$$x^r=a^r+ra^{r-1}(x-a)+r(r-1)\xi_a^{r-2}\frac{(x-a)^2}{2} \tag{2}$$

where ξ_a is between x and a. Set $x=x_i>0$ in (2), multiply by $\alpha_i>0$ and sum for $i=1,\ldots,n$. Without loss of generality let us assume that $\Sigma_{i=1}^n\alpha_i=1$ and let us use the notation $\Sigma\beta_i$ for $\Sigma_{i=1}^n\beta_i$. Set $a=\Sigma\alpha_i x_i>0$ in (2). It follows that

$$\sum\alpha_i x_i^r\left\{\begin{matrix}\geq\\ \leq\end{matrix}\right\}\left(\sum\alpha_i x_i\right)^r \quad \text{for } \begin{cases} r<0 \quad \text{or} \quad r>1,\\ 0<r<1.\end{cases} \tag{3}$$

Equality holds in (3) iff the x_i's are all equal. Relation (3) shows that $\phi(x)=x^r$ is strictly convex (concave) for $r<0$ or $r>1$ ($0<r<1$).

Using the increasing or decreasing property of $\phi(x)=x^r$ we obtain from (3)

$$\sum\alpha_i x_i\left\{\begin{matrix}\leq\\ \geq\end{matrix}\right\}\left(\sum\alpha_i x_i^r\right)^{1/r} \quad \text{for } \begin{cases} r>1,\\ r<1, \quad r\neq 0,\end{cases} \tag{4}$$

with equality iff the x_i's are all equal.