We have not nearly described all the classes of functions that answer our original question; many other classes exist. Students who discover, generalize, and classify such vector space functions are likely to develop a deeper understanding of linear transformations, and will gain an appreciation of the open-ended nature of mathematical research.

Reference

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On "Rethinking Rigor in Calculus...," or Why We Don't Do Calculus on the Rational Numbers

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In a recent "Point/Counterpoint" in the *American Mathematical Monthly* ([**1**], [**2**]), it was suggested that the basic theorems on continuous functions and their derivatives (the Boundedness Theorem, the Extreme Value Theorem, the Intermediate Value Theorem, and, especially, the Mean Value Theorem) be omitted from the introductory calculus course. Reasons given were that "the origin of the Mean Value Theorem in the structure of the real numbers ... is too difficult for a standard course"; that these discussions are "the sort of thing that gives mathematics a bad name: assuming the nonobvious to prove the obvious"; that perhaps there is no "need for formal theorems and proofs in a standard calculus course"; and that, in any event, one shouldn't "prove things in more generality than is necessary; even analysts don't usually deal with the discontinuous derivatives allowed by the Mean Value Theorem."

I demur. Without commenting on the pedagogical issues, I would like to point out that this program risks serious misdirection of the mathematical intuition of its students. In particular, I submit that the notion that these basic theorems are "obvious," save for obscure subtleties raised only by bizarre, pathological functions (which are scarcely encountered in practice) is incorrect.

A quick glance at the standard proofs of these basic theorems on continuous functions shows that they represent direct (or nearly direct) applications of the Axiom of Completeness as applied to their *domain*—that is, they reflect the existence of particular limit points guaranteed by the Axiom of Completeness, acting on the domain of a continuous, real-valued function. One way to see what is going on is to consider continuous functions on an *incomplete* domain, say the set of rational numbers, $\mathbf{Q}$.

Of course, it is important to remember that continuity depends only on the points where a function is defined — that is to say, on the points *in the domain* of the function. Many of the examples that follow have been chosen to highlight the "hole" in the rational number line at $1/\sqrt{2}$, in recognition of the historic role of $\sqrt{2}$ as perhaps the first number shown to be irrational. Note that it is possible to encounter this difficulty even without explicit mention of the irrational number $\sqrt{2}$.

To begin with, a non-degenerate, closed, bounded interval of rational numbers is not a compact set (that is, the Heine-Borel theorem fails over **Q**):

$$\text{Let } U = \left\{x \in [0,1] \cap \mathbf{Q} \mid x^2 < \tfrac{1}{2}\right\},$$

$$\text{Let } V = \left\{x \in [0,1] \cap \mathbf{Q} \mid x^2 > \tfrac{1}{2}\right\};$$

Then the open cover $\{[0,u),(v,1];\ u \in U,\ v \in V\}$ of $[0,1]$ in **Q** does not have a finite subcover.

Theorems that depend on the compactness of a closed, bounded interval thus also fail over **Q**. For example, a function continuous on a closed, bounded interval of rational numbers need not be bounded. For example, consider $f(x) = 1/(2x^2-1)^2$, which is continuous on $[0,1] \cap \mathbf{Q}$. This function (see Figure 1) also serves as a counterexample to the theorem (valid over **R**) that a function, continuous over a closed, bounded interval, is uniformly continuous there.

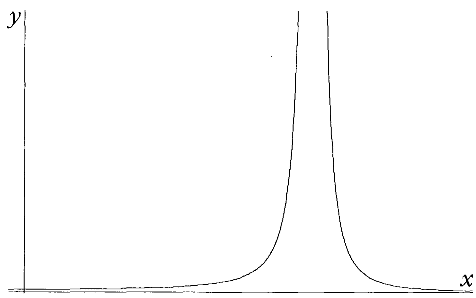

Figure 1. $f(x) = 1/(2x^2-1)^2$.

The Extreme Value Theorem is usually deduced by yet another application of the Axiom of Completeness, once the initial fact of boundedness has already been deduced. Clearly, the Extreme Value Theorem fails if the function is not itself bounded (as seen in the example above), but it may fail even in the case of a bounded function continuous on a closed, bounded interval of **Q**: Consider (see Figure 2) the function $f(x) = 1-(2x^2-1)^2$, which fails to attain its supremum of 1 on the domain $[0,1] \cap \mathbf{Q}$.

Figure 2. $f(x) = 1-(2x^2-1)^2$.

The Intermediate Value Theorem is a direct reflection of the fact that a closed interval of real numbers is connected. (The theorem can be proved by "hammer and tongs" without mentioning connectedness, but such proofs only amount to a recapitulation of the proof of connectedness in the more specific context.) On the other hand, a non-degenerate closed interval in **Q** is not connected. For example the sets U and V given in the first example constitute a disconnection of the interval $[0,1] \cap \mathbf{Q}$. It is thus no surprise that the intermediate value theorem fails over **Q**: consider the function $f(x) = 1/(2x^2 - 1)$, which is continuous on $[0,1] \cap \mathbf{Q}$, satisfies $f(0) = -1$, $f(1) = 1$, but fails to satisfy $f(x) = 0$ anywhere on $[0,1] \cap \mathbf{Q}$ (see Figure 3).

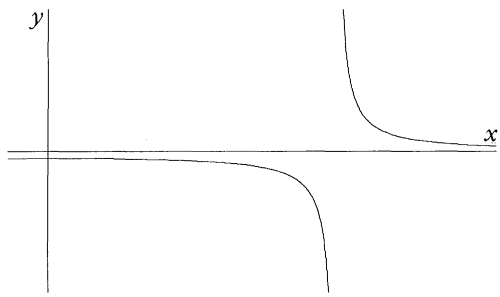

Figure 3. $f(x) = 1/(2x^2 - 1)$.

Indeed, the Intermediate Value theorem may fail over **Q** even for a polynomial: for example, the polynomial function $f(x) = x^2 - 2$ satisfies $f(0) = -2$, $f(2) = +2$, but there is no rational number x for which $x^2 - 2 = 0$.

Finally, Rolle's theorem, and hence the Mean Value Theorem, fails on **Q** as well. For example (see Figure 4), the function $f(x) = 1 - \sqrt{4x^4 - 4x^2 + 1}$ is continuous on $[0,1] \cap \mathbf{Q}$ and differentiable on $(0,1) \cap \mathbf{Q}$; indeed the derivative is continuous at every point of $(0,1) \cap \mathbf{Q}$. Though $f(0) = f(1) = 0$, there is no x where $f'(x) = 0$.

Even the watered-down "Increasing Function Theorem," proferred in **[1]** as a more sincere replacement for the Mean Value Theorem, fails over **Q**: consider (see

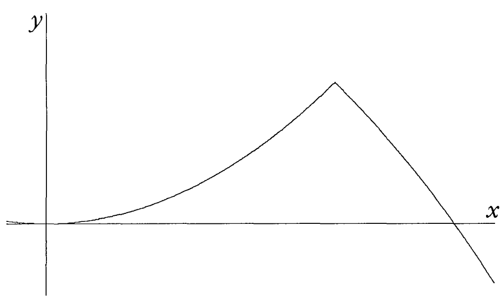

Figure 4. $f(x) = 1 - \sqrt{4x^4 - 4x^2 + 1}$.

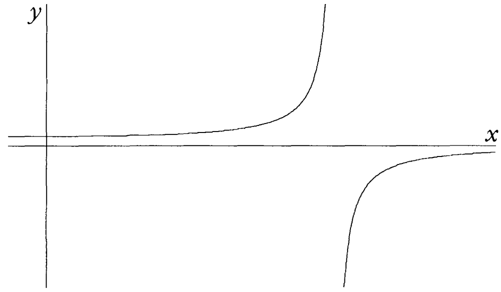

Figure 5. $f(x) = \dfrac{1}{1 - 2x^2}$.

Figure 5) the function $f(x) = 1/(1 - 2x^2)$ over the interval $[0, 1] \cap \mathbf{Q}$. The derivative is positive and even continuous on $[0, 1] \cap \mathbf{Q}$, but the function is not increasing.

There are no functions here more exotic than polynomials, rational expressions, and square roots—surely functions dealt with regularly by "analysts" and even ordinary users of the calculus. The familiar theorems on continuous functions fail, not because there is anything pathological about them, but because the domain has been changed from **R**, which is complete, to **Q**, which is not. These examples suggest that an attempt to separate the familiar properties of continuous functions from the underlying structure of the real line is unlikely to succeed.

The great theorems of the calculus are not necessarily "obvious"—otherwise it would not have taken nearly 2,000 years of mathematical effort to discover them or their proofs. To hide from our students the persuasive arguments by which we have come to believe them is to do them a disservice.

References

1. T. W. Tucker, Rethinking rigor in calculus, *Amer. Math. Monthly* 104 (1997), 231–240.
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A Far-reaching Formula

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It is well known that the formula for the area of a trapezoid—the average length of the parallel sides times the perpendicular distance between them—gives the areas of squares, rectangles, parallelograms, and triangles. In Figure 1, h is the distance between the parallel lines l and m.

It is not so well known that the formula can be used to find other areas. In Figure 2, the length of the radius of the circle is similar to h in Figure 1. We can consider the center and the circle to be the parallel sides so, as in Figure 3, the circumference of the circle is the base of the triangle and the area of the circle is, by the trapezoid formula, $(2\pi r + 0)r/2 = \pi r^2$.