

Lattices of Trigonometric Identities

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This capsule illustrates how a standard calculus exercise, the evaluation of

$$\int \sin x \cos x \, dx, \quad (1)$$

can lead to some interesting combinatorial identities. Evaluating (1) by the substitution $u = \sin x$ yields $\sin^2 x / 2 + C_1$, whereas the evaluation of (1) via $u = \cos x$ yields $-\cos^2 x / 2 + C_2$. Since these antiderivatives differ only by a constant, we obtain $\sin^2 x + \cos^2 x = C$ for some constant C . Setting $x = 0$ yields

$$\sin^2 x + \cos^2 x = 1. \quad (2)$$

Now suppose we proceed analogously with

$$\int \sin^{2m+1} x \cos^{2n+1} x \, dx, \quad (3)$$

where both m and n are nonnegative integers. As with (1), we have at hand two complementary methods for the evaluation of (3). Since (1) enabled us to rediscover (2), we may expect that (3) will lead to some generalization of that basic identity.

Suppose we first split off a power of $\cos x$, replace $\cos^{2n} x$ by $(1 - \sin^2 x)^n$, and let $u = \sin x$. Then (3) reduces to

$$\int u^{2m+1} (1 - u^2)^n \, du.$$

Then use of the binomial expansion yields

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \int u^{2m+2i+1} \, du = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{\sin^{2m+2i+2} x}{2m+2i+2} + C_1,$$

which we write as

$$\frac{\sin^{2m+2} x}{2} \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{\sin^{2i} x}{m+i+1} + C_1. \quad (3a)$$

On the other hand, we can compute (3) by reversing the roles of $\sin x$ and $\cos x$ (which, in turn, reverses the roles of m and n) and letting $u = \cos x$. The result is

$$-\frac{\cos^{2n+2} x}{2} \sum_{i=0}^m (-1)^i \binom{m}{i} \frac{\cos^{2i} x}{n+i+1} + C_2. \quad (3b)$$

Since (3a) and (3b) differ only by a constant,

$$(\sin^{2m+2} x) \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{\sin^{2i} x}{m+i+1} + (\cos^{2n+2} x) \sum_{i=0}^m (-1)^i \binom{m}{i} \frac{\cos^{2i} x}{n+i+1} = C_{m,n} \quad (4)$$

for some constant $C_{m,n}$. It may be interesting to interpret (4) as a “first-quadrant integer lattice of trigonometric identities” generalizing the fundamental relation (2) with $C_{0,0} = 1$.

One would now like to know the value of $C_{m,n}$ as a function of m and n . Substituting $x = 0$ in (4) yields

$$C_{m,n} = \sum_{i=0}^m (-1)^i \binom{m}{i} \frac{1}{n+i+1},$$

whereas the substitution $x = \pi/2$ yields

$$C_{m,n} = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{1}{m+i+1}.$$

Hence,

$$\sum_{i=0}^m (-1)^i \binom{m}{i} \frac{1}{n+i+1} = C_{m,n} = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{1}{m+i+1}. \quad (5)$$

This demonstrates that $C_{m,n}$ and the expression

$$\sum_{i=0}^m (-1)^i \binom{m}{i} \frac{1}{n+i+1}$$

are both symmetric in m and n .

It may be instructive to attempt an explanation of this symmetry. Recall that the Beta function is symmetric in its two arguments:

$$B(m+1, n+1) = \int_0^1 x^m (1-x)^n dx. \quad (6)$$

Upon expanding $(1-x)^n$, we find that (6) becomes

$$B(m+1, n+1) = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{1}{m+i+1} = C_{m,n}.$$

So now we can evaluate $C_{m,n}$ using the standard relationship

$$B(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

[$\Gamma(k) = (k-1)!$ for positive integers k] between the Beta function $B(x, y)$ and the Gamma function $\Gamma(x)$. This leads directly to

$$C_{m,n} = \frac{m!n!}{(m+n+1)!}, \quad (7)$$

a form in which the symmetry of $C_{m,n}$ is explicit.

An appealing form for $C_{m,n}$ is acquired by taking $x = \pi/4$ in (4):

$$\begin{aligned} C_{m,n} = & \left(\frac{1}{2}\right)^{m+1} \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{1}{2^i(m+i+1)} \\ & + \left(\frac{1}{2}\right)^{n+1} \sum_{i=0}^m (-1)^i \binom{m}{i} \frac{1}{2^i(n+i+1)}. \end{aligned} \quad (8)$$

Here the symmetry of $C_{m,n}$ is again apparent. Now let $m = n$ in both (5) and (8).

Equating the resulting expressions for $C_{n,n}$ produces the striking identity

$$\sum_{i=0}^n (-1)^i \binom{n}{i} \frac{1}{n+i+1} = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{1}{2^{n+i}(n+i+1)}. \quad (9)$$

Since

$$C_{n,n} = \frac{(n!)^2}{(2n+1)!} = \frac{1}{(2n+1)\binom{2n}{n}},$$

we can use the expressions in (9) to obtain formulas for the reciprocals of $\binom{2n}{n}$, the central binomial coefficients.

Finally, it is worth noting that a similar analysis may be performed on

$$\int \tan^{2m+1} x \sec^{2n+2} x dx,$$

using the substitutions $u = \tan x$ and $u = \sec x$. This results in

$$(\tan^{2m+2} x) \sum_{i=0}^n \binom{n}{i} \frac{\tan^{2i} x}{m+i+1} + (-1)^{m+1} (\sec^{2n+2} x) \sum_{i=0}^m (-1)^i \binom{m}{i} \frac{\sec^{2i} x}{n+i+1} = C'_{m,n}, \quad (10)$$

a lattice of identities that both generalizes $\sec^2 x - \tan^2 x = 1$ and reflects its asymmetry. Substituting $x = 0$ in (10) yields

$$C'_{m,n} = (-1)^{m+1} \sum_{i=0}^m (-1)^i \binom{m}{i} \frac{1}{n+i+1}, \quad (11)$$

and comparing with (5) gives $C'_{m,n} = (-1)^{m+1} C_{m,n}$. Now substituting $x = \pi/4$ in (10) yields

$$C'_{m,n} = \sum_{i=0}^n \binom{n}{i} \frac{1}{m+i+1} + 2^{n+1} (-1)^{m+1} \sum_{i=0}^m (-2)^i \binom{m}{i} \frac{1}{n+i+1}. \quad (12)$$

Choosing $m = n$ in both (11) and (12) and equating the resulting expressions produces the identity

$$\sum_{i=0}^n (-1)^{n+i+1} \frac{\binom{n}{i}}{n+i+1} = \sum_{i=0}^n [1 + (-2)^{n+i+1}] \frac{\binom{n}{i}}{n+i+1}.$$

The fact that there is nothing deep or elegant in any of this only adds to the charm of these investigations.

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