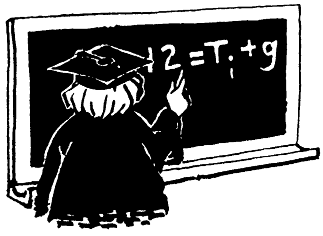

CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

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Arithmetic Progressions and the Consumer

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After the review of a homework problem requested by one of the students in a technical mathematics course, it was belatedly discovered that there were two versions of Allyn Washington's 3rd edition *Basic Technical Mathematics with Calculus*, one in metric and the other not. Murphy's Law and the natural reticence of students being what they are, this did not become apparent until after the problem in the wrong edition had been solved. The mistake proved serendipitous, however, because the two seemingly parallel problems produced results whose difference seemed to fly in the face of common sense, and ended up creating a lively discussion, which led to some new insights on arithmetic progressions.

Problem 1. A well-driller charges \$3 for drilling the first foot of a well, and for every foot thereafter he charges one cent more than the preceding foot. How much does he charge for drilling a 500-ft well?

Problem 2 (*metric*). A well-driller charges \$9 for drilling the first metre of a well, and for every metre thereafter he charges three cents more than the preceding metre. How much does he charge for drilling a 200-m well?

Except for the depths of the wells, the problems were obviously intended to represent approximately equivalent conditions (for our purposes, we will read "yard" for "metre" in Problem 2). However, when one solved the problems, using the formula

$$S(a,d,n) = na + (n)(n-1)d/2 \tag{1}$$

for the sum of the first n terms of an arithmetic progression having first term a and common difference d, it turned out to cost considerably more for the shallower well.

The 200-yd well cost 200(9) + (200)(199)(.03)/2 = \$2397, whereas the 500-ft well cost (500)(3) + (500)(499)(.01)/2 = \$2747.50. More cost less!

Several theories were suggested for the discrepancy, but it was only when we looked more closely at the nonmetric version that it became apparent that the one-cent increase for each additional foot produced an increment of nine cents for each additional yard, not the three-cent increment that seemed proper:

first yard:	\$3.00 + 3.01 + 3.02 = \$9.03
second yard:	\$3.03 + 3.04 + 3.05 = \$9.12
third yard:	\$3.06 + 3.07 + 3.08 = \$9.21.

Each foot in the yard costs three cents more than the corresponding foot in the previous yard, so that the whole yard will indeed cost nine cents more. The cumulative effect of the three-cent increment is a critical concern. To make the problems more nearly equivalent, the author should have required that the second well-digger charge \$9.03 for the first yard and nine cents for each additional yard. The message here to the consumer is: be careful of your units for pricing; they can work for you or against you.

Now let us consider a more general setting. Suppose we have two arithmetic progressions, one having m terms and the other having $n = km$. To see when $S(a, d, km) = S(\alpha, \delta, m)$, we proceed as above, grouping the terms in the longer sequence into blocks of length $k = n/m$:

$$\begin{aligned} a + (a+d) + (a+2d) + \cdots + (a+(k-1)d) &= S \\ (a+kd) + (a+(k+1)d) + (a+(k+2)d) + \cdots + (a+(2k-1)d) &= S + k^2 d \\ (a+2kd) + (a+(2k+1)d) + (a+(2k+2)d) + \cdots + (a+(3k-1)d) &= S + 2k^2 d \\ &\vdots \\ (a+(m-1)kd) + (a+[(m-1)k+1]d) + \cdots + (a+(mk-1)d) &= S + (m-1)k^2 d. \end{aligned}$$

Since each term in a block is kd more than the corresponding term in the preceding one, and since there are k such terms in each block, the common difference between the sums of blocks is $k^2 d$. Note, by (1), that the first block sums to $S = ka + k(k-1)d/2$. Therefore, upon summing both sides of the above array, we obtain

$$S(a, d, km) = m(ka + k(k-1)d/2) + m(m-1)k^2 d/2.$$

Accordingly, $S(a, d, km)$ equals $S(\alpha, \delta, m)$ if and only if

$$ka + k(k-1)d/2 + (m-1)k^2 d/2 = \alpha + (m-1)\delta/2. \tag{2}$$

There are many solutions to (2), one of which is

$$\delta = k^2 d \qquad \alpha = ka + k(k-1)d/2. \tag{3}$$

For instance, consider $a = 1$, $d = 2$, $m = 3$, $k = 4$. Then, from (2), we have $\alpha + \delta = 48$ and $S(1, 2, 12) = S(x, 48-x, 3)$ for every natural number x.

Returning to our well-drilling problem, we noted earlier that a 200-yd well costing \$9 for drilling the first yard and an increase of 3 cents in price for each subsequent yard would cost $S(9, .03, 200) = \$2397$. To obtain this cost for the same type of incremental pricing policy in terms of feet, instead of yards, we need to determine a and d so that $S(a,d,600) = S(9,.03,200)$. Using (2), with $k = 3$, we have

$$a + 299.5d = 3.995.$$

Thus, there are many pricing solutions to our problem. Using (3), for example, we obtain $d = .03/9 = 1/300$ and $a = 3 - 1/300 = 2.99\overline{6}$. In particular, one solution would be to pay \$$2.99\overline{6}$ for drilling the first foot and 1/3 cent more for each additional foot.

The class can now see how practical mathematics may be. Indeed, from (2) and (3), we obtain

$$S\big(\alpha/k - (k-1)\delta/2k^2, \delta/k^2, km\big) = S(\alpha, \delta, m) \tag{4}$$

and

$$S(a,d,km) = S\big(ka + k(k-1)d/2, k^2 d, m\big). \tag{5}$$

If your contractor offers to use units which are smaller by a factor of $1/k$ (e.g., feet instead of yards) so "you only will pay for what you use," then you must make sure that the price for the first unit is reduced to $\alpha/k - (k-1)\delta/2k^2$ and that the price for each additional unit is reduced by a factor of $1/k^2$. On the other hand, if your contractor insists on using units that are larger by a factor of k "to reduce his overhead," while only increasing his pricing increment by the same factor, by all means do it. The savings $m^2k(k-1)d/2$ may more than pay for your subscription to this journal.

More Applications of the Mean Value Theorem

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The Mean Value Theorem applied to $f(x) = \log x$ on $[a,b]$ yields $\log b - \log a = (b-a)(1/c)$ for some $c \in (a,b)$. This can be recast as

$$(b-a)/b < \log b/a < (b-a)/a \tag{1}$$

in order to obtain efficient proofs of the following:

(i) $n^m > m^n$ if $m > n \geqslant e$ and $n^m < m^n$ if $e \geqslant m > n > 0$.

(ii) $\sqrt[n]{a_1 a_2 \cdots a_n} \leqslant (1/n)\sum_{i=1}^{n} a_i$ (a_i positive) with equality holding if and only if $a_1 = a_2 = \cdots = a_n$.

(iii) $(1+1/n)^n < e < (1+1/n)^{n+1}$ for all positive integers n.

To establish (i), let $a = n$ and $b = m$. If $n \geqslant e$, the the right-hand inequality in (1) yields

$$(m/n)^n < e^{m-n} \leqslant n^{m-n},$$

from which $m^n < n^m$ follows. If $e \geqslant m$ and $n > 0$, the left-hand inequality in (1) yields

$$m^{(m-n)/m} \leqslant e^{(m-n)/m} < m/n.$$