The Pen and the Barn

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A standard differential calculus problem involves showing that the rectangle with fixed perimeter and maximal area is a square. Textbooks often present this in terms of a farmer who wishes to build the most efficient rectangular pen with a fixed length of fencing. A variant uses one wall of a barn for all or part of one side of the pen (Figure 1). Let's take a new look at this latter class of problems, because there is an attractive way to show how the lengths of the fence and the barn wall affect the shape of the optimal pen.

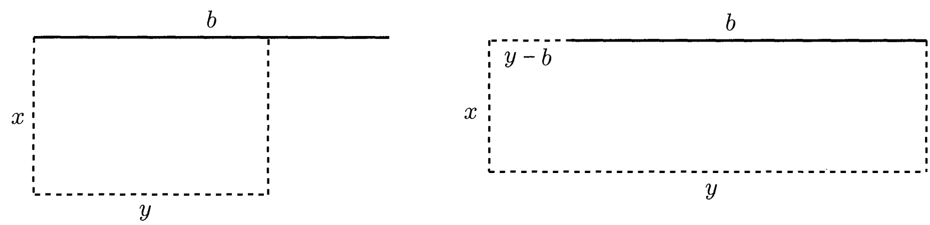

Figure 1. Solid lines indicate barn wall.

Suppose that the barn wall is b feet long, and let t denote the length of fencing available. We will hold b constant but let the parameter t vary as we explore the different cases. The claim is that the ratio r of the length y to the width x of the optimal pen as a function of t is given by the piecewise expression

$$r(t) = \begin{cases} 2, & \text{if } t \leq 2b \\ 2b/(t-b), & \text{if } 2b \leq t \leq 3b \\ 1, & \text{if } 3b \leq t \end{cases} \tag{1}$$

(also see Figure 2). That is, the optimal pen is a square if the length of fencing t is at least three times the length b of the barn wall; but as t decreases from $3b$, the ratio of sides for the optimal pen increases to 2. What is the significance of the boundary points $t = 2b$ and $t = 3b$? And why are there three regions of t values yet only two different configurations of pens?

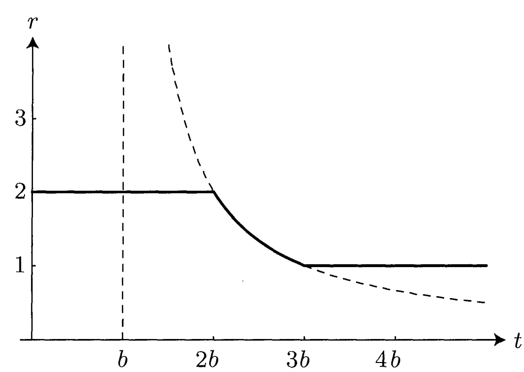

Figure 2

Case 1. Say the barn wall forms one entire side of the pen as on the left in Figure 1. Let the pen have length y (parallel to the barn) and width x. Then $y + 2x = t$ and the area of the pen is $A = xy = x(t - 2x)$ for the domain $(t-b)/2 \le x \le t/2$. The lower bound on x follows from the assumption that $y \le b$. Now $A'(x) = t - 4x$ so the maximal area occurs either at $x = t/4$, if this is in the domain, or else at the endpoint $x = (t-b)/2$. To summarize, the optimal pen using the barn wall as one entire side is described by

$$x = \frac{t}{4},\ y = \frac{t}{2} \qquad \text{if } \frac{t-b}{2} \le \frac{t}{4} \quad (\text{that is, } t \le 2b);\ \text{or}$$
$$x = \frac{t-b}{2},\ y = b \qquad \text{if } \frac{t}{4} \le \frac{t-b}{2} \quad (2b \le t).$$

Case 2. Say the barn wall forms only a part of one side of the pen, as on the right in Figure 1. Since $y \ge b$ the pen extends $y - b$ beyond the corner of the barn, and $y + 2x + (y - b) = t$. The area of the pen is now given by $A = xy = x(t + b - 2x)/2$ for the domain $0 \le x \le (t-b)/2$. The upper bound on x is derived from $y \ge b$. Since $A'(x) = (t+b)/2 - 2x$, the maximal area occurs either at $x = (t+b)/4$, if this is in the domain, or at the endpoint $x = (t-b)/2$. Thus the optimal pen when the barn is only a part of one side is described by

$$x = \frac{t-b}{2},\ y = b \qquad \text{if } \frac{t-b}{2} \le \frac{t+b}{4} \quad (\text{that is, } t \le 3b);\ \text{or}$$
$$x = \frac{t+b}{4},\ y = \frac{t+b}{4} \qquad \text{if } \frac{t+b}{4} \le \frac{t-b}{2} \quad (3b \le t).$$

All that remains is to compare the areas of optimal pens obtained by the two cases. But it is already apparent that if $2b \le t \le 3b$ then the two cases yield the same result: a pen with ratio $y/x = 2b/(t-b)$ whose maximal area occurs at an endpoint of the domain in each case. It is easy to see now that (1) correctly describes the ratio y/x of the dimensions of the optimal pen for each value of t.

The intermediate situation in which $2b < t < 3b$ is not often addressed in calculus texts. For example, if $t = \sqrt{5}b$, then the optimal pen forms a golden rectangle.

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Exploiting a Factorization of $x^n - y^n$

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Early in elementary algebra classes students factor $x^2 - y^2$, $x^3 - y^3, \ldots$, where x, y represent real numbers. Later they learn that

$$x^n - y^n = (x - y) \sum_{m=0}^{n-1} x^m y^{n-1-m} \tag{1}$$

for each positive integer n and all real x, y.