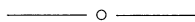


Since $\sqrt{2} > 4/\pi > \sqrt{3/2}$, the conjecture about the heights is validated for very small values of Δ/L .

We have seen three different models for the buckling of the rail, each using a different mathematics for its formulation. For a given length and extension, each may be solved using a computer algebra system. The dimensionless form of the solution is $h/L = \sqrt{\Delta/L} g(\Delta/L)$ where g is an analytic function of its argument. We have, therefore, given the leading term of the series for g . In many instances, a formula reveals relations that are not readily apparent in a numerical solution. In these three problems, for example, we see that the buckled height is proportional to the geometric mean of L and Δ .



Who Cares If $X^2 + 1 = 0$ Has a Solution?

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The shortest path between two truths in the real domain passes through the complex domain.—*Jacques Hadamard*

Most mathematics textbooks introduce complex numbers as a means of solving equations that obviously have no real solutions. A typical introduction goes something like this:

The equation $x^2 + 1 = 0$ has no real solution because there is no real number x that can be squared to produce -1 . To solve such an equation, mathematicians created an expanded system of numbers using the imaginary unit i , defined as $i = \sqrt{-1}$.

A student may well ask: Why solve this equation in the first place? And in any case, who cares if it has a solution?

These are legitimate questions. One would expect a practical or intuitive justification for introducing such a novel idea. After all, there are direct and intuitive motivations for introducing other aspects of our number system. The natural numbers are used for counting, negative numbers may be used to describe debt, rational numbers help us describe such natural concepts as “half a quart of milk,” and irrational numbers are needed for representing certain distances in the plane. On the other hand, there is no easy application of complex numbers that serves to motivate their use at the usual introductory level. Moreover, by the time students are sophisticated enough to understand the applications of complex numbers, the need to motivate them is usually forgotten.

In this paper we give four situations that can serve to motivate complex numbers for students who have had two semesters of calculus. We have found that the best motivation for most new ideas is their utility in solving real problems. The examples presented here use complex numbers as a tool for obtaining real answers in real situations.

The mother of invention. Historically, complex numbers were introduced for practical reasons. Their use by Rafael Bombelli (1526–1572) provides insight into the need for complex numbers.

In the sixteenth century mathematicians were interested in finding solutions (real, of course) of polynomial equations. One of the high points [3] was Cardano’s solution

of the cubic equation $x^3 + ax^2 + bx + c = 0$. The substitution $x = y - (a/3)$ eliminates the quadratic term, yielding a “depressed” equation of the form

$$y^3 + py + q = 0.$$

Thus it is sufficient to consider depressed cubic equations.

The substitution $y = u - (p/3u)$ gives $u^6 + qu^3 - (p^3/27) = 0$. By the quadratic formula, we get

$$u^3 = \frac{1}{2} \left(-q \pm \sqrt{q^2 + (4p^3/27)} \right),$$

which gives us u . Substituting this value of u into the expression for y , and making use of the fact that $p^3/27u^3 = u^3 + q$, we obtain the *cubic formula*:

$$y = \sqrt[3]{\frac{-q \pm \sqrt{q^2 + (4p^3/27)}}{2}} - \sqrt[3]{\frac{q \pm \sqrt{q^2 + (4p^3/27)}}{2}}.$$

Bombelli applied this formula to the equation $y^3 - 15y - 4 = 0$, obtaining

$$y = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}}.$$

But it turns out that this solution is just $y = 4$.

To see this, observe that a cube root of $2 + \sqrt{-121}$ is $2 + \sqrt{-1}$. Bombelli discovered this (through trial and error) by cubing $2 + \sqrt{-1}$ and treating the square of $\sqrt{-1}$ as -1 . Similarly, a cube root of $-2 + \sqrt{-121}$ is $-2 + \sqrt{-1}$. It follows that the solution of the equation is $(2 + \sqrt{-1}) - (-2 + \sqrt{-1}) = 4$.

The point is, Cardano’s formula gives the correct answer 4, but to obtain it from the formula, we had to apply algebraic rules to expressions such as $\sqrt{-1}$. This was Bombelli’s motivation for accepting complex numbers: They helped him obtain *real* solutions to cubic equations.

A “unifying” concept. A fundamental physical problem is to describe the motion of a mass m suspended from a spring with spring constant k . The mathematical model for this mechanical system is the differential equation

$$my'' + ky = 0,$$

where $y(t)$ is the position of the mass at time t relative to its equilibrium position [1]. For simplicity we assume that $m = k = 1$ and that the mass has initial position $y(0) = 1$ and initial velocity $y'(0) = 0$.

It is usual to assume solutions of the form $y = e^{rt}$, which upon substitution into the differential equation leads to the characteristic equation $r^2 + 1 = 0$. But this latter equation has the complex roots i and $-i$, so we get the complex solution $y = c_1 e^{it} + c_2 e^{-it}$. Now, making use of the initial conditions, we obtain $c_1 + c_2 = 1$ and $ic_1 - ic_2 = 0$, so $c_1 = c_2 = \frac{1}{2}$. Thus, $y = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it}$. But then $y = \cos t$, a *real* solution to the real physical problem!

One may ask: Why not assume a solution of the form $A \cos t + B \sin t$ to begin with? Of course we can do that, but by using complex numbers we can solve all homogeneous linear differential equations with constant coefficients in a unified setting—namely, we may always assume solutions of the form $y = e^{rt}$.

Indeed, it may not be so easy to guess the form of the solution for a damped mechanical system such as

$$y'' + 2y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

However, by simply assuming a solution of the form $y = e^{rt}$ we get the characteristic equation $r^2 + 2r + 2 = 0$, with complex roots $1 + i$ and $1 - i$. This gives solutions of the form

$$y = c_1 e^{(1+i)t} + c_2 e^{(1-i)t}.$$

The initial conditions now yield the real solution $y = e^t \cos t + e^t \sin t$.

For efficiency. If one already has a knowledge of complex numbers, then an easy way of evaluating $\int e^{ax} \cos bx \, dx$ is to consider

$$\int e^{(a+bi)x} \, dx = \frac{1}{a+bi} e^{(a+bi)x} = \frac{a-bi}{a^2+b^2} e^{ax} (\cos bx + i \sin bx).$$

The integral we want to evaluate is simply the real part of the last expression; that is,

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx).$$

Insight from the complex view. The function $f(x) = 1/(1+x^2)$ has the power series representation

$$f(x) = 1 - x^2 + x^4 - x^6 + \dots$$

This series is obtained from the geometric series for $1/(1-x)$ by substituting $-x^2$ for x . The function f is very well behaved; it is infinitely differentiable and bounded. So one may expect that the series for f would converge for all x . But the ratio test shows that the radius of convergence of this series is 1. What stops the convergence beyond 1 and -1 ?

To answer this, we can look at f as a function of a complex variable. The singularities of $f(z) = 1/(1+z^2)$ are i and $-i$, which are a distance 1 from the origin. So the domain of convergence of the complex series

$$f(z) = 1 - z^2 + z^4 - z^6 + \dots$$

is the unit disc D [2]. Since the interval of convergence I of the corresponding real series is the restriction of D to the real line, it follows that $I = (-1, 1)$.

We thus explain the behavior of a *real* function by looking at its complex twin.

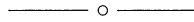
Conclusion. It is intriguing how complex numbers naturally occur in solving real problems. These few examples are a mere aperitif to the many deep insights that complex numbers provide. It is interesting and instructive to find other situations where complex numbers can be used to solve or explain a real problem.

One of the most basic observations about complex numbers is that they complete our understanding of our number system in an elegant way: A polynomial equation with complex coefficients must have complex solutions. Complex numbers even

help explain why the product of two negative real numbers must be positive (recall that multiplication of complex numbers involves a rotation). Indeed, the shortest path between two truths in the real domain does pass through the complex domain.

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Polishing the Star

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Recently, Hoehn proved the following interesting theorem about a pentagram [A Menelaus-type theorem for the pentagram, *Mathematics Magazine* 66:2 121–123] Hoehn used Menelaus’ theorem 20 times, but it is possible to give a much simpler proof. Geometry students may enjoy seeing results concerning the pentagram as an application of the Law of Sines.

Theorem. *If $A_1B_1A_2B_2A_3B_3A_4B_4A_5B_5$ is a pentagram (see Figure 1), then*

$$\frac{A_1B_1}{B_1A_2} \cdot \frac{A_2B_2}{B_2A_3} \cdot \frac{A_3B_3}{B_3A_4} \cdot \frac{A_4B_4}{B_4A_5} \cdot \frac{A_5B_5}{B_5A_1} = 1 \tag{1}$$

and

$$\frac{B_1A_3}{A_3B_4} \cdot \frac{B_4A_1}{A_1B_2} \cdot \frac{B_2A_4}{A_4B_5} \cdot \frac{B_5A_2}{A_2B_3} \cdot \frac{B_3A_5}{A_5B_1} = 1. \tag{2}$$

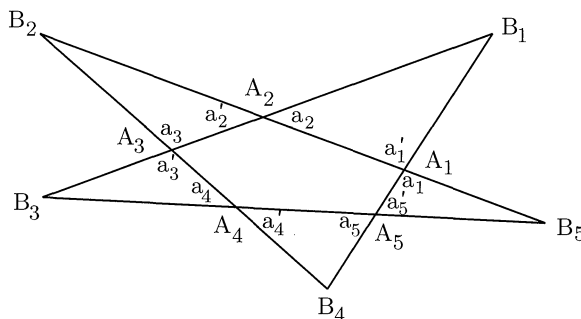


Figure 1