$$r(\vec{x}, \vec{y}) = \frac{n \sum x_i y_i - \left(\sum x_i\right)\left(\sum y_i\right)}{\sqrt{\sum x_i{}^2 - \left(\sum x_i\right)^2}\sqrt{n \sum y_i{}^2 - \left(\sum y_i\right)^2}}$$

$$= \frac{y_n \left\{ \frac{n \sum_{i=1}^{n-1} x_i y_i}{y_n} + n x_n \right\} - y_n \left\{ \left(\sum x_i\right) \left(\frac{\sum_{i=1}^{n-1} y_i}{y_n} + 1 \right) \right\}}{\sqrt{\sum x_i{}^2 - \left(\sum x_i\right)^2}\; y_n \sqrt{\frac{n\left(\sum_{i=1}^{n-1} y_i{}^2\right)}{y_n{}^2} + n - \left(\frac{\sum_{i=1}^{n-1} y_i}{y_n} + 1 \right)^2}}$$

and observing that

$$\lim_{y_n \to \infty} r(\vec{x}, \vec{y}) = \frac{n x_n - \sum x_i}{\sqrt{\sum x_i{}^2 - \left(\sum x_i\right)^2}\sqrt{n-1}} = r(\vec{x}, \vec{y}_n(0, 1)).$$

Recall that in our first classroom example, the correlation was independent (except for sign) of y_6, as long as $y_1 = y_2 = \cdots = y_5$. Now we see that if y_6 tends to infinity, its value dominates the expression for r so much as to make the other y values irrelevant, even when they aren't all the same!

Exact Values for the Sine and Cosine of Multiples of 18°—A Geometric Approach

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Questions involving the exact values of the trigonometric functions of 18°, 36°, 54° and 72° have been popular in compilations of challenging mathematical problems, and in the problems sections of mathematics journals. (See, for example, problems 189 and 339 in [**1**] and the sections entitled Trigonometry: numerical evaluation, Trigonometry: numerical identities and Trigonometry: theory of equations in [**2**].) Invariably, the determination of these values make little or no use of geometry; rather, the derivations rely heavily upon trigonometric identities, such as the double and triple angle formulas or the product-to-sum formulas. In this note, we will use a purely geometric approach to derive the values of the sine and cosine of multiples of 18°. Therefore, this capsule can be used as enrichment or investigative material for students in trigonometry or precalculus courses.

Since 0° and 90° are already standard trigonometric angles, we consider only angles of measure 18°, 36°, 54° and 72°. Using Figure 1, we first show that the sine and cosine of 18°, 36°, 54° and 72° can all be expressed as functions of x. Toward this end, draw the altitude CD from vertex C to the side AB (Figure 1(b)), and let $\overline{BD} = y$. For the right triangle BCD, we find (by the Pythagorean theorem) that $\overline{CD}^2 = x^2 - y^2$, and for the right triangle ADC, we see that $\overline{CD}^2 = 1 - (1-y)^2 = 2y - y^2$. Equating the expressions for $\overline{CD}^2$ leads to $y = \frac{1}{2}x^2$. Therefore,

$$\overline{AD} = 1 - \frac{1}{2}x^2, \quad \overline{BD} = \frac{1}{2}x^2, \quad \text{and} \quad \overline{CD} = x\sqrt{1 - \frac{1}{4}x^2},$$

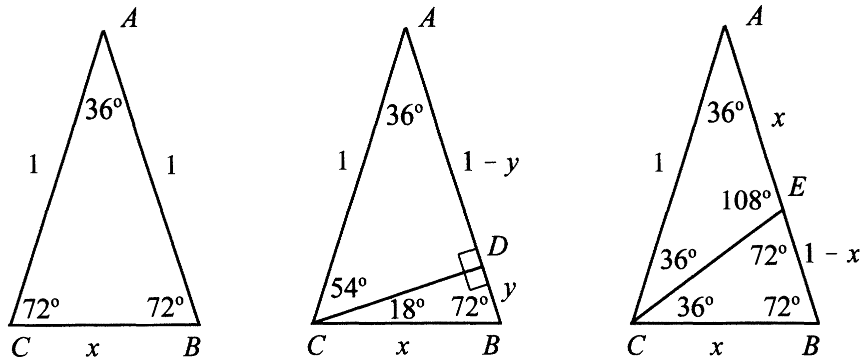

Figure 1(a). **Figure 1(b).** **Figure 1(c).**

and we have the formulas

$$\cos 18^\circ = \sin 72^\circ = \sqrt{1 - \frac{1}{4}x^2} \qquad \cos 36^\circ = \sin 54^\circ = 1 - \frac{1}{2}x^2$$

$$\cos 54^\circ = \sin 36^\circ = x\sqrt{1 - \frac{1}{4}x^2} \qquad \cos 72^\circ = \sin 18^\circ = \frac{1}{2}x.$$

To determine the value of x, let CE be the angle bisector of angle C in Figure 1. Then (Figure 1(c)) $\triangle BCE$ and $\triangle AEC$ are isosceles triangles. Since $\triangle ABC$ is similar to $\triangle CEB$,

$$\frac{\overline{AB}}{\overline{CB}} = \frac{\overline{CB}}{\overline{BE}} \quad \text{or} \quad \frac{1}{x} = \frac{x}{1-x}.$$

The equation $x^2 = 1 - x$ has solutions $\frac{-1+\sqrt{5}}{2}$ and $\frac{-1-\sqrt{5}}{2}$, which means that in Figure 1(c) the length of a side is $\frac{-1+\sqrt{5}}{2}$. Therefore, our formulas yield the exact values

$$\cos 18^\circ = \sin 72^\circ = \frac{\sqrt{5+\sqrt{5}}}{2\sqrt{2}} \qquad \cos 36^\circ = \sin 54^\circ = \frac{\sqrt{5}+1}{4}$$

$$\cos 54^\circ = \sin 36^\circ = \frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}} \qquad \cos 72^\circ = \sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

References

1. Edward J. Barbeau, Murray S. Klamkin and William O. J. Moser, *Five Hundred Mathematical Challenges*, Mathematical Association of America, 1995.
2. Stanley Rabinowitz, ed., *Index of Mathematical Problems: 1980–1984*, MathPro Press, 1992.