Differentials and Elementary Calculus

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The purpose of this capsule is to provide an example illustrating the use of the differential in basic calculus. The commonly used examples seem badly out of date when our students either own or have ready access to computers and sophisticated calculators. Examples of the type given below will make the topic more relevant and meaningful as well as giving new meaning and importance to the topic of implicit differentiation. The method is well known and is studied in most elementary courses in differential equations, namely the Euler "tangent line" method.

In the typical calculus text one finds $df = f'(x)\,dx$ and $\Delta f \doteq df$, for well-behaved functions, and thus

$$f(x+h) \doteq f(x) + f'(x)h. \tag{1}$$

To illustrate the utility of equation (1) let us consider the equation

$$\sin x \cos y = y. \tag{2}$$

This equation defines y implicitly as a function of x. Using implicit differentiation we have

$$y' = \frac{\cos x \cos y}{1 + \sin x \sin y}. \tag{3}$$

Most students are uncomfortable at this point because they feel the underlying function $y(x)$ is somehow less real than functions that can be exhibited explicitly. They think it is less real because the defining rule is too complex; they would prefer of course to be able to solve for y! To combat this idea I usually encourage my students to attempt to find at least one point on the curve defined by a given equation. Sometimes this is very difficult but in the present case students can determine after a while that $(0,0)$ is a point on the curve defined by equation (2). It is then possible to lead them to approximate $y(1)$, $y(2.3)$, or what have you as we show below.

Combining (1) and (3) we obtain

$$y(x+h) \doteq y(x) + h\left(\frac{\cos x \cos y(x)}{1 + \sin x \sin y(x)}\right). \tag{4}$$

Thus taking $h = 0.05$ in (4) and recalling that $y(0) = 0$ we find

$$y(0.05) \doteq y(0) + 0.05\left(\frac{\cos 0 \cos 0}{1 + \sin 0 \sin 0}\right) = 0.05.$$

Of course at this stage of the game students still question how valuable our ability to approximate values of y really is. This questioning disappears only when they see that we can continue our procedure to approximate $y(x)$ for x far away from 0. For instance, since we have deduced above that $(0.05, 0.05)$ is very nearly on the curve we seek, we have

$$y(0.1) \doteq y(0.05) + 0.05\left(\frac{\cos 0.05 \cos 0.05}{1 + \sin 0.05 \sin 0.05}\right)$$

or $y(0.1) \doteq 0.0997508$, and we can continue this process as long as we like. The approximate values of $y(x)$ for $x = 0, 0.05, 0.1, 0.15, \ldots, 0.95, 1$ are shown in Table 1.

Table 1

x	$y(x)$
0.0	0.0
0.05	0.05
0.10	0.0997508
0.15	0.1487664
0.20	0.1965994
0.25	0.2428634
0.30	0.2872467
0.35	0.3295173
0.40	0.3695203
0.45	0.4071697
0.50	0.4424365
0.55	0.4753372
0.60	0.5059219
0.65	0.5342638
0.70	0.5604510
0.75	0.5845799
0.80	0.6067498
0.85	0.6270597
0.90	0.6456055
0.95	0.6624783
1.0	0.6777631

To obtain a rough feeling for the accuracy of an approximation one should not, of course, go into a rigorous error analysis. Students will readily believe that accuracy depends on the size of h and one can reinforce this belief as follows. Table 1 shows $y(1) \doteq 0.6777631$ and equation (2) asserts that, were our values of y exact, we would have

$$\sin 1 \cos 0.6777631 = 0.6777631.$$

In fact

$$\sin 1 \cos 0.6777631 = 0.6554868$$

which means our approximation is not very accurate. If however one uses $h = 0.01$ in equation (4) the approximation given for $y(1)$ is 0.6660564 and

$$\sin 1 \cos 0.6660564 = 0.6616187$$

indicating that this approximation can be very accurate indeed.

This example illustrates one way to proceed when teaching differentials. There are many benefits to be derived from introducing Euler's method at the time one introduces the differential. Perhaps most important is the simple fact that students are convinced differentials have some utility. In addition students learn an important technique which will reappear in later courses in a more sophisticated form. Finally, the computer may be worked into the elementary calculus in a significant way.