because each entry in Pascal's Triangle is the sum of the two entries that "straddle" it in the previous row, and so the process of taking successive differences starting on a *diagonal* leads back to the appropriate *row*. The entries in the $i$th diagonal of Pascal's Triangle, then, are given by

$$d_i(k) = \frac{k(k+1)(k+2)(k+3)\cdots(k+i-1)}{i!}. \tag{2}$$

Notice the striking similarity of equations (1) and (2): the only difference is that the signs within each factor are reversed. If those factors are multiplied out for each value of $i$, the coefficients of the resulting polynomials are the entries in the rows of the Factorial Triangle.

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*Editor's Note*: Readers interested in a fuller exposition on sequences generated by polynomials may enjoy Calvin Long's article "Pascal's Triangle, Difference Tables, and Arithmetic Sequences of Order $n$," CMJ 15 (September 1984) 290–298.

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## Finding Bounds for Definite Integrals

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Students in elementary calculus are often dismayed to learn that not every function has an antiderivative, and consequently not every definite integral can be evaluated by the Fundamental Theorem. Although most textbooks discuss such things as Simpson's Rule and the Trapezoid Rule, these methods are usually long and tedious to apply. In many cases, reasonably good bounds for definite integrals can be obtained with little effort by the use of well-known theorems. The fact that techniques for doing this have never been discussed in one place is the motivation for this note.

Except for very specialized and esoteric results, the following three theorems provide methods for obtaining such bounds.

**Theorem A.** *If $f$, $g$, and $h$ are integrable and satisfy $g(x) \leqslant f(x) \leqslant h(x)$ on the interval $[a,b]$, then*

$$\int_a^b g(x)\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b h(x)\,dx.$$

**Theorem B.** *On the interval $[a,b]$, suppose that $f$ and $g$ are integrable, $g$ never changes sign, and $m \leqslant f(x) \leqslant M$. Then*

$$m\int_a^b g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx \leqslant M\int_a^b g(x)\,dx.$$

**Theorem C.** *If $f$ and $g$ are integrable on $[a,b]$, then*

$$\int_a^b f(x)g(x)\,dx \leqslant \sqrt{\int_a^b f^2(x)\,dx}\;\sqrt{\int_a^b g^2(x)\,dx}\;.$$

*Example* 1. Find bounds for $\int_1^2 \frac{x\,dx}{\sqrt{x^3+8}}$. Using Theorem B, we choose $f(x) = \frac{1}{\sqrt{x^3+8}}$ and $g(x) = x$. Then $\frac{1}{4} \leqslant f(x) \leqslant \frac{1}{3}$ for $x \in [1,2]$, and $\int_1^2 g(x)\,dx = \frac{3}{2}$. Therefore,

$$.375 \leqslant \int_1^2 \frac{x\,dx}{\sqrt{x^3+8}} \leqslant .500. \tag{1}$$

If we use Theorem C, it is natural to choose $f(x) = \frac{x}{\sqrt{x^3+8}}$ and $g(x) = 1$. It follows that

$$\int_1^2 \frac{x\,dx}{\sqrt{x^3+8}} \leqslant \sqrt{\int_1^2 \frac{x^2\,dx}{x^3+8}}\ \sqrt{\int_1^2 1^2\,dx} = \sqrt{\frac{2}{3}\ln\frac{4}{3}} < .438, \tag{2}$$

a considerable improvement over the upper bound obtained in (1). Since $x^2 \leqslant x^3 \leqslant x^4$ on $[1,2]$, Theorem A yields

$$\int_1^2 \frac{x\,dx}{\sqrt{x^4+8}} \leqslant \int_1^2 \frac{x\,dx}{\sqrt{x^3+8}} \leqslant \int_1^2 \frac{x\,dx}{\sqrt{x^2+8}}.$$

The integral on the right equals $2\sqrt{3} - 3$, while the one on the left equals $\frac{1}{2}\ln\left(\frac{2+\sqrt{6}}{2}\right)$. Thus,

$$.399 < \int_1^2 \frac{x\,dx}{\sqrt{x^3+8}} < .465. \tag{3}$$

Combining (1), (2), and (3), we obtain

$$.399 < \int_1^2 \frac{x\,dx}{\sqrt{x^3+8}} < .438. \tag{4}$$

Note that in each of the three theorems, we were not forced into the choices actually made. Other possibilities exist, resulting in different bounds.

In using Theorem B, one interprets a given integrand as the product of two functions $f$ and $g$, where $g$ is of one sign and can easily be integrated. The calculation of $m$ and $M$ is usually straightforward. One's inclination, of course, is to let $g$ be something easy to integrate. There is sometimes more than one reasonable "decomposition," as the following example shows.

*Example* 2. For the integral $\int_0^1 x^2 e^{-x^2}\,dx$, we can let $f(x) = e^{-x^2}$ and $g(x) = x^2$. Then $\frac{1}{e} \leqslant f(x) \leqslant 1$ on $[0,1]$ and $\int_0^1 g(x)\,dx = \frac{1}{3}$. Hence,

$$\frac{1}{3e} \leqslant \int_0^1 x^2 e^{-x^2}\,dx \leqslant \frac{1}{3}.$$

Other choices of $f$ and $g$ exist, however. Since $xe^{-x^2}$ can easily be integrated, let $g(x) = xe^{-x^2}$ and $f(x) = x$. Then $0 \leqslant f(x) \leqslant 1$ on $[0, 1]$ and $\int_0^1 g(x)\,dx = (e-1)/2e$ yield

$$0 \leqslant \int_0^1 x^2 e^{-x^2}\,dx \leqslant \frac{e-1}{2e} < .316. \tag{5}$$

What if $g(x) = x$ and $f(x) = xe^{-x^2}$? Then $\int_0^1 g(x)\,dx = \frac{1}{2}$, and $f$ has its minimum value $m = 0$ and maximum value $M = 1/\sqrt{2e}$. Consequently,

$$0 \leqslant \int_0^1 x^2 e^{-x^2}\,dx \leqslant \frac{1}{\sqrt{8e}}\,. \tag{6}$$

A quick check with the calculator shows that the upper bound in (6) is substantially better than in (4) and (5). Combining the best of all cases,

$$.122 < \frac{1}{3e} \leqslant \int_0^1 x^2 e^{-x^2}\,dx \leqslant \frac{1}{\sqrt{8e}} < .215. \tag{7}$$

If Theorem A is used on this example, we might reason that $x^2 \leqslant x$ on $[0, 1]$, and this leads to the inequality

$$\int_0^1 x^2 e^{-x}\,dx \leqslant \int_0^1 x^2 e^{-x^2}\,dx.$$

The integral on the left equals $2 - (5/e) > .160$, giving us a better lower bound than in (7). Consequently,

$$.160 < \int_0^1 x^2 e^{-x^2}\,dx < .215.$$

*Example* 3. Find bounds for $\int_0^{\pi/3} x\,dx/\cos x$. Using Theorem B, the obvious decompositions do not yield particularly good results. Suppose, however, that $f(x) = x/\sin x$ (with $f(0)$ taken to be 1) and $g(x) = \sin x/\cos x$. Then $f$ has $m = 1$ and $M = 2\pi/3\sqrt{3}$, and so

$$.693 < \ln 2 \leqslant \int_0^{\pi/3} \frac{x\,dx}{\cos x} \leqslant \frac{2\pi}{3\sqrt{3}} \ln 2 < .839. \tag{8}$$

Another estimate can also be obtained from Theorem A. Since $1 - x^2/2 \leqslant \cos x \leqslant 1$,

$$\int_0^{\pi/3} x\,dx \leqslant \int_0^{\pi/3} \frac{x\,dx}{\cos x} \leqslant \int_0^{\pi/3} \frac{x\,dx}{1 - \frac{1}{2}x^2}\,.$$

The lower bound here (.548) is worse than that in (8), but the value of the right-hand integral is $\ln\left(\dfrac{18}{18 - \pi^2}\right)$. Hence,

$$.693 < \int_0^{\pi/3} \frac{x\,dx}{\cos x} < .795.$$

Students find these methods a welcome change of pace from the routine numerical techniques. They enjoy the challenge to improve on bounds already obtained, and they gain valuable experience in working with inequalities, the heart of analysis.

**Right Triangles with Perimeter and Area Equal**

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Students learning about right triangles may observe that the area equals the perimeter for the 6, 8, 10 and for the 5, 12, 13 right triangles. The question naturally occurs whether this situation also holds for other right triangles whose legs have integral length. Thus, the following discussion may be of interest.

If $a, b$ and $\sqrt{a^2+b^2}$ are the sides of a right triangle, then the perimeter is $a+b+\sqrt{a^2+b^2}$ and the area is $ab/2$. Equating both expressions yields

$$a^2+b^2=\frac{a^2b^2}{4}-ab^2-a^2b+2ab+a^2+b^2,$$

which reduces to

$$a+b=\frac{ab}{4}+2. \qquad (*)$$

Since $a$ and $b$ are natural numbers, it follows that $ab/4$ is a natural number and thus either 4 divides $a$ or $b$, or 2 divides both $a$ and $b$. In the latter case, letting $a=2p$ and $b=2q$, we find that $2p+2q=pq+2$. Therefore, 2 divides $p$ or $q$, and so either $a$ or $b$ is divisible by 4. (Another elementary, but instructive, approach is the following: if neither $a$ nor $b$ is divisible by 4, then both $a$ and $b$ are even. Therefore, $(a/2)(b/2)=ab/4=(a+b)-2$ is even. But then either $a/2$ or $b/2$ is divisible by 2, and this contradicts the assumption that neither $a$ nor $b$ was divisible by 4.)

Suppose that $b=4m$ for some positive integer $m$. Substituting in $(*)$, we get $a+4m=am+2$. Since this can be written as

$$(a-4)+2=(a-4)m,$$

we see that $(a-4)$ divides 2. Thus, the only solutions to our problem occur for $a=5$ and $a=6$, with respective values $b=12$ and $b=8$.

Having come this far, instructors can introduce primitive Pythagorean triangles and raise the following conjecture:

*For every natural number n, there is at least one primitive Pythagorean triangle in which the area equals n times the perimeter.*

The case $n=1$ yields the above cited 5, 12, 13 triangle. Now we may be motivated to try to verify this conjecture or to read the proof of Problem 3587 in *School Science and Mathematics* 76 (1976) 83–84.