length of the chord determined by the horizontal line $y = 2b$, where b is the ordinate of the vertex. The case for $f(x)$ concave down is analogous.

To verify the interpretation of α and β in Figure 1, let $f(x) = c(x - a)^2 + b$ for $b, c > 0$. Then f has roots $a \pm i\sqrt{b/c}$, and $\alpha = a$ and $\beta = \sqrt{b/c}$ are precisely the asserted lengths.

This interpretation seems natural because in the case of real roots the picture (Figure 2) is the same, except that now the horizontal line is merely the x-axis itself; the roots are $\alpha \pm \beta$ instead of $\alpha \pm i\beta$, where $\beta = \sqrt{|b/c|}$.

We can now see both of these pictures unified in a simple bifurcation process: imagine a parabola (say, $b < 0 < c$) moving uniformly in the positive y-direction, and observe the behavior of the roots in the complex plane. As b increases toward 0, we see that $\beta = \sqrt{|b/c|}$ decreases toward 0. Thus, as long as the parabola meets the x-axis, the roots are real, symmetric about $x = a$, and are converging toward each other (Figure 3a). At the bifurcation point, $b = 0$ and the roots are coincident (Figure 3b). As b increases above 0, β increases and the two roots diverge in the imaginary direction (Figure 3c) at the same rate as their previous motion in Figure 3a.

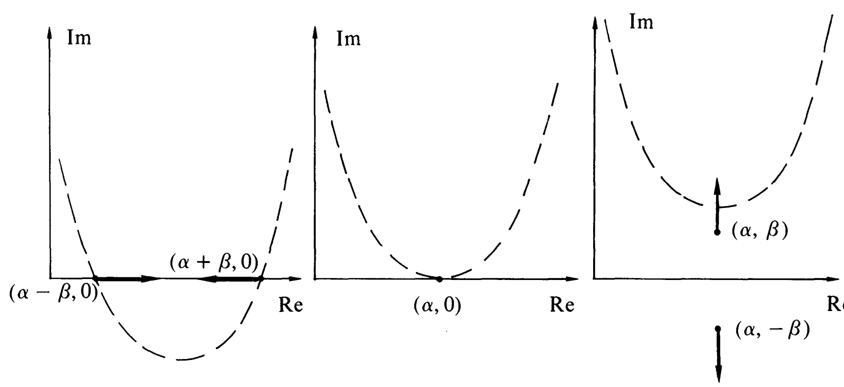

Figure 3a. Figure 3b. Figure 3c.

Another Look at $x^{1/x}$

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Here is a nice way for students to apply and combine two important results. First, we use the inequality between the arithmetic and geometric means to prove that the sequence $\sqrt[n]{n}$ tends to 1 as $n \to \infty$. Then the Mean Value theorem for derivatives will be used to show that the maximum value of $x^{1/x}$ occurs at $x = e$.

According to the arithmetic-geometric mean inequality,

$$\sqrt[n]{a_1 a_2 \cdots a_n} < \frac{a_1 + a_2 + \cdots + a_n}{n}$$

when the positive numbers $a_1, a_2, \ldots, a_n$ are not all equal. Suppose we set $a_1 = a_2 = \cdots = a_{n-1} = 1$ and let $a_n = \sqrt{n}$. Then, since $\sqrt[n]{\sqrt{n}} > 1$, we get

$$1 < \sqrt[n]{\sqrt{n}} < \frac{n-1+\sqrt{n}}{n} < 1 + \frac{1}{\sqrt{n}}.$$

Squaring gives

$$1 < \sqrt[n]{n} < 1 + \frac{2}{\sqrt{n}} + \frac{1}{n},$$

and thus $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Now suppose $a > e$. Then, by the Mean Value theorem,

$$\frac{\ln a - \ln e}{a - e} = \frac{1}{c} \qquad \text{for some} \quad c \in (e, a).$$

Hence,

$$\frac{\ln a - \ln e}{a - e} < \frac{1}{e} \quad \text{or} \quad \ln a - 1 < \frac{a}{e} - 1.$$

This gives $\ln a^{1/a} < \ln e^{1/e}$. Accordingly,

$$a^{1/a} < e^{1/e} \qquad \text{for all} \quad a > e. \tag{$*$}$$

(It may be interesting to observe that for $a = \pi$, we obtain the familiar inequality $\pi^e < e^\pi$.) For $0 < b < e$, the Mean Value theorem yields

$$\frac{\ln e - \ln b}{e - b} = \frac{1}{c'} \qquad \text{for some} \quad c' \in (b, e).$$

Hence,

$$\frac{1 - \ln b}{e - b} > \frac{1}{e} \quad \text{or} \quad \frac{1}{e}\ln e > \frac{1}{b}\ln b.$$

In particular,

$$b^{1/b} < e^{1/e} \qquad \text{for all} \quad b < e. \tag{$**$}$$

Combining ($*$) and ($**$), we see that $x^{1/x}$ has its maximum value at $x = e$.

The preceding results may serve as motivation for students to corroborate—with a little more information obtained from the derivative of $x^{1/x}$—that the curve $y = x^{1/x}$ has the graph

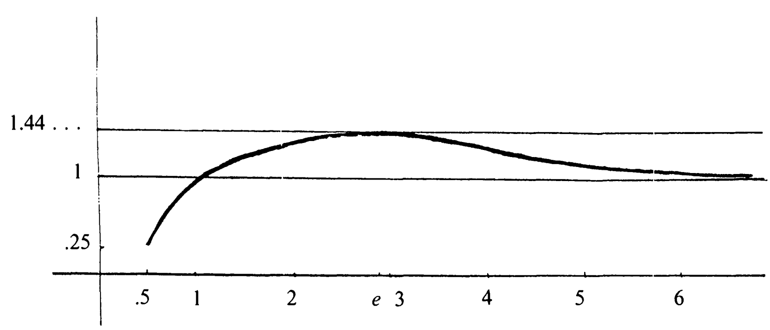

Figure 1.