Some Sums of Some Significance

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After the first year of calculus, any of our students can tell us[1] that

$$\sum_{k=0}^{\infty} \frac{1}{2^k} = 2$$

or that

$$\sum_{k=0}^{\infty} \frac{1}{k!} = e,$$

but practically any other infinite sums, such as the closely related

$$\sum_{k=0}^{\infty} \frac{k^n}{2^k} \tag{1}$$

or

$$\sum_{k=0}^{\infty} \frac{k^n}{k!} \tag{2}$$

are regarded as yes-or-no questions, as in: "Yes; by the ratio test." It turns out that exact values for these sums are very easy to obtain. In [**3**] Alan Gorfin derived a recurrence relation that can be used to compute (1) for any chosen n. In this note we do the same for (2) and then show that the two sums have closely related combinatorial interpretations and that both recurrences can be easily derived using intuitive counting arguments.

Let

$$M_n = \sum_{k=0}^{\infty} \frac{k^n}{k!}. \tag{3}$$

Then $M_0 = e$ (providing we take the term $0^0/0!$ to be 1), $M_1 = \sum_{k=0}^{\infty} k/k! = \sum_{k=1}^{\infty} k/k! = \sum_{k=1}^{\infty} 1/(k-1)! = \sum_{k=0}^{\infty} 1/k! = e$, and using the binomial theorem,

$$\begin{aligned} M_n &= \sum_{k=0}^{\infty} \frac{k^n}{k!} = \sum_{k=1}^{\infty} \frac{k^n}{k!} = \sum_{k=1}^{\infty} \frac{k^{n-1}}{(k-1)!} \\ &= \sum_{k=0}^{\infty} \frac{(k+1)^{n-1}}{k!} = \sum_{k=0}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{k^j}{k!} \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} M_j. \end{aligned} \tag{4}$$

[1] Well, perhaps this is wishful thinking.

Thus

$$M_2 = \binom{1}{0} M_0 + \binom{1}{1} M_1 = 2e,$$
$$M_3 = \binom{2}{0} M_0 + \binom{2}{1} M_1 + \binom{2}{2} M_2 = 5e,$$

and by repeated use of (4) one can compute any M_n and, indeed, $\sum_k [p(k)]/k!$ for any polynomial p.

Gorfin [**3**] derives a similar relation for

$$S_n = \sum_{k=0}^{\infty} \frac{k^n}{r^k}, \tag{5}$$

where $r > 1$, showing that

$$S_n = \frac{1}{r-1} \sum_{j=0}^{n-1} \binom{n}{j} S_j.$$

We next show how the recurrences for S_n and M_n arise in a combinatorial context. If we have a set of n distinct objects and j identical boxes, where $0 \le j \le n$, the total number of ways of selecting one object to put in each box is the familiar binomial coefficient $\binom{n}{j}$. But suppose instead we want to distribute *all* the objects among the j boxes, leaving none of them empty; that is, we want to partition the n elements into j nonempty subsets. The number of ways of doing this is known as a *Stirling number of the second kind*, sometimes denoted $\left\{ {n \atop j} \right\}$ (see exercises 3 and 4). Just as the total number of subsets of an n-element set is $\sum_{j=0}^{n} \binom{n}{j} = 2^n$, the sum

$$B_n = \sum_{j=0}^{n} \left\{ {n \atop j} \right\} \tag{6}$$

represents the total number of ways of partitioning an n-element set into nonempty subsets. The numbers B_n are generally known as *Bell numbers*, to honor E. T. Bell.

It is not difficult to derive a recurrence for B_n. Note that $B_0 = 1$ (there is exactly one way of partitioning an empty set). Given $n \ge 1$ objects $a_1, a_2, \ldots, a_n$, we first put a_n in a box; then for each j, $0 \le j \le n-1$, we can choose j additional objects to go in the box with a_n, and we can do so in $\binom{n-1}{j}$ ways. The remaining $n-1-j$ objects can then be partitioned in B_{n-1-j} ways. Summing over all j, the total value is given by

$$\begin{aligned} B_n &= \sum_{j=0}^{n-1} \binom{n-1}{j} B_{n-1-j} = \sum_{j=0}^{n-1} \binom{n-1}{n-1-j} B_{n-1-j} \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} B_j. \end{aligned}$$

The numbers B_n thus satisfy the same recurrence as the numbers M_n of (3). Since $B_0 = \frac{1}{e} M_0$, it follows that $B_n = \frac{1}{e} M_n$ for all n.

Now consider the same situation—partitioning an n-element set into j nonempty subsets—where in addition the j subsets are distinguishable, for instance, we might regard each subset as a box labeled with a number 1 through j. There are $\left\{ {n \atop j} \right\}$

ways to partition the objects into j nonempty groups, but now the groups may be permuted among the boxes in $j!$ ways, yielding $j!\left\{{n \atop j}\right\}$ possible arrangements. Such a distribution has been called a *preferential arrangement* [**5**], since it amounts to a distribution of all the objects into j different ranks or preference groups. Let

$$P_n = \sum_{j=0}^{n} j!\left\{{n \atop j}\right\},$$

the total number of preferential arrangements of n objects. To derive a recurrence for P_n, initially $P_0 = 1$; when $n > 0$ there is at least one box, so we may assume one of the boxes is labeled "#1." Then for each j, $1 \le j \le n$, there are $\binom{n}{j}$ ways to choose j objects to go into box #1, and then P_{n-j} preferential arrangements of the remaining $n - j$ objects, so

$$\begin{aligned} P_n &= \sum_{j=1}^{n} \binom{n}{j} P_{n-j} = \sum_{j=1}^{n} \binom{n}{n-j} P_{n-j} \\ &= \sum_{j=0}^{n-1} \binom{n}{j} P_j, \end{aligned} \tag{7}$$

which is the same recurrence satisfied by the numbers S_n of (5) with $r = 2$. Since $P_0 = \frac{1}{2}S_0$, we have $P_n = \frac{1}{2}S_n$ for all n.

Both problems considered here have long histories. The curious reader might start with the 1859 paper by Arthur Cayley [**1**], where a derivation of the recurrence (7) is given. A different derivation, in connection with the sum (1), is found in [**5**]. The recurrence for B_n as well as the sum (3) appeared in [**2**] in 1887, and [**6**] contains extensive further references on the numbers B_n. The textbook *Concrete Mathematics* [**4**] is an outstanding survey of useful connections between *con*tinuous and dis*crete* mathematics.

Exercises.

1. Find the exact value of $\sum_{j=0}^{\infty}(3k^3 - k^2 + 2)/k!$.
2. Show that $\sum_{k=0}^{\infty} k/(k+1)! = 1$.
3. Prove that, for $n > 0$ and $1 < j < n$, $\left\{{n \atop j}\right\} = j\left\{{n-1 \atop j}\right\} + \left\{{n-1 \atop j-1}\right\}$. (*Hint:* Given a set $\{a_1, \ldots, a_n\}$, consider the partitions of $\{a_1, \ldots, a_{n-1}\}$; there are two cases, depending on what is done with a_n.)
4. Define $\left\{{0 \atop 0}\right\} = 1$, and note that $\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$ and $\left\{{n \atop 0}\right\} = 0$ for $n > 0$. Use exercise 3 to construct the first few rows of a table of values for $\left\{{n \atop j}\right\}$ similar to Pascal's triangle.

References

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2. Maurice d'Ocagne, Sur une classe de nombres remarquables, *American Journal of Mathematics* 9 (1887) 353–380.

3. A. Gorfin, Evaluating the sum of the series $\sum_{k=1}^{\infty} k^j/M^k$, *College Mathematics Journal* 20 (1989) 329–331.
4. Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics: A Foundation For Computer Science*, Addison-Wesley, Reading, MA, 1989.
5. O. A. Gross, Preferential arrangements, *American Mathematical Monthly* 69 (1962) 4–8.
6. Gian-Carlo Rota, The number of partitions of a set, *American Mathematical Monthly* 71 (1964) 498–503.

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A Rose Is a Rose Is a Rose ...

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In this computer and writing project, second-semester calculus students use computer graphics or calculator graphics to examine graphs of the family of n-leaved roses and some variations. Students get the valuable experience of "tweaking" an equation a bit and explaining how that affects the graph. Before I assign the project, we discuss graphing (by hand) in polar coordinates, as well as finding areas bounded by curves in polar coordinates. The class has two weeks to work on the project and students are encouraged to consult one another, although each student must hand in an individual report. In grading the project, I give the most credit to logical conclusions and clear explanations.

Problem 1. Plot the polar curves

$$r = \cos(nt) \qquad \text{and} \qquad r = \sin(nt)$$

for $n = 1, 2, \ldots, 5$. (See Figure 1.) Use the smallest interval of t values over which the polar graph is the complete curve. From your observations, make some conjectures about the curves (number of petals, angles where the petals are centered, etc.). Try to account for the difference between odd n and even n.

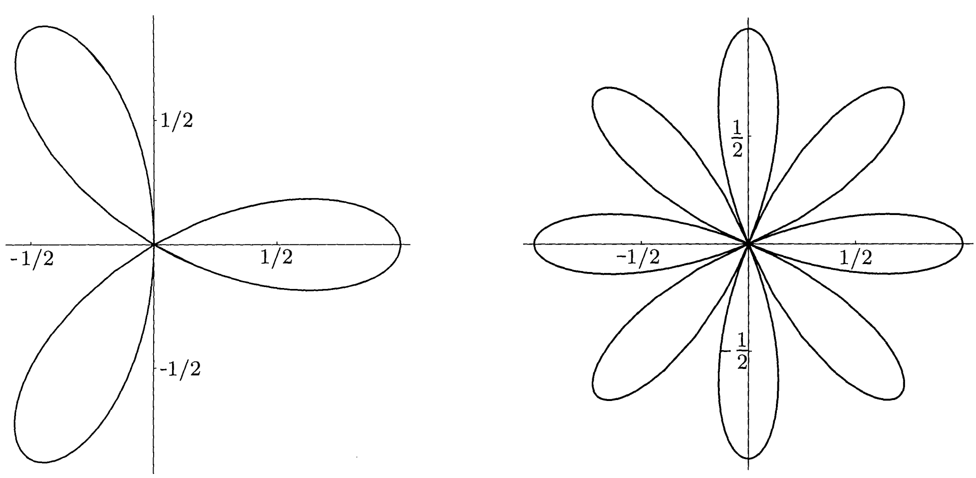

Figure 1. Left: $r = \cos(3\theta)$. Right: $r = \cos(4\theta)$.