In this octant, the points of P_n satisfy (1) and

$$x_1 + x_2 + \cdots + x_k - x_{k+1} - x_{k+2} - \cdots - x_n + |x_1 + x_2 + \cdots + x_n| \leqslant 2.$$

Equivalently (as can be easily verified by considering the two cases $\sum_{i=1}^{n} x_i \geqslant 0$ or $\leqslant 0$),

$$x_i \geqslant 0 \quad (1 \leqslant i \leqslant k) \quad \text{and} \quad x_i \leqslant 0 \quad (k+1 \leqslant i \leqslant n),$$

$$\sum_{i=1}^{k} x_i \leqslant 1 \quad \text{and} \quad \sum_{i=k+1}^{n} x_i \geqslant -1.$$

In order to compute the content of this part of P_n we let $u_i = -x_{k+i}$ $(1 \leqslant i \leqslant n-k)$ and compute the volume of the congruent polytope whose points $(x_1, x_2, \ldots, x_k, u_1, u_2, \ldots, u_{n-k})$ satisfy:

$$x_i \geqslant 0 \quad (1 \leqslant i \leqslant k) \quad \text{and} \quad u_i \geqslant 0 \quad (1 \leqslant i \leqslant n-k),$$

$$\sum_{i=1}^{k} x_i \leqslant 1 \quad \text{and} \quad \sum_{i=1}^{n-k} u_i \leqslant 1.$$

This volume is given by

$$\int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-\sum_{i=1}^{k-1} x_i} \int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-\sum_{i=1}^{n-k-1} u_i} du_{n-k} \cdots du_2\, du_1\, dx_k \cdots dx_2\, dx_1$$

which (à la Hsu) evaluates to $(1/(n-k)!)(1/k!)$. Thus, the part of P_n in the $\binom{n}{k}$ octants (each with k nonnegative and $n-k$ nonpositive coordinates) is

$$\binom{n}{k} \frac{1}{k!\,(n-k)!} = \frac{1}{n!} \binom{n}{k} \binom{n}{n-k}$$

and the content of P_n is

$$\sum_{k=0}^{n} \frac{1}{n!} \binom{n}{k} \binom{n}{n-k} = \frac{1}{n!} \binom{2n}{n},$$

where this last equality is simply Vandermonde's identity. [See, for example, John Riordan's *An Introduction to Combinatorial Analysis*, Wiley and Sons (1958)15.]

The Derivatives of Sin x and Cos x

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In this note, we offer simple proofs of the formulas $\frac{d}{dx}(\sin x) = \cos x$ and $\frac{d}{dx}(\cos x) = -\sin x$ for x acute. Both derivations rest on the figure below and avoid the necessity of first deriving $\lim_{h \to 0} \frac{\sin h}{h} = 1$. They also do not require the use of the formula for the difference of two sines or the formula for the sine of the sum of two angles followed by the formula for $1 - \cos x$ in terms of half angles. One of these is usually part of the standard treatment for evaluating $\lim_{\Delta x \to 0} \dfrac{\sin(x + \Delta x) - \sin x}{\Delta x}$.

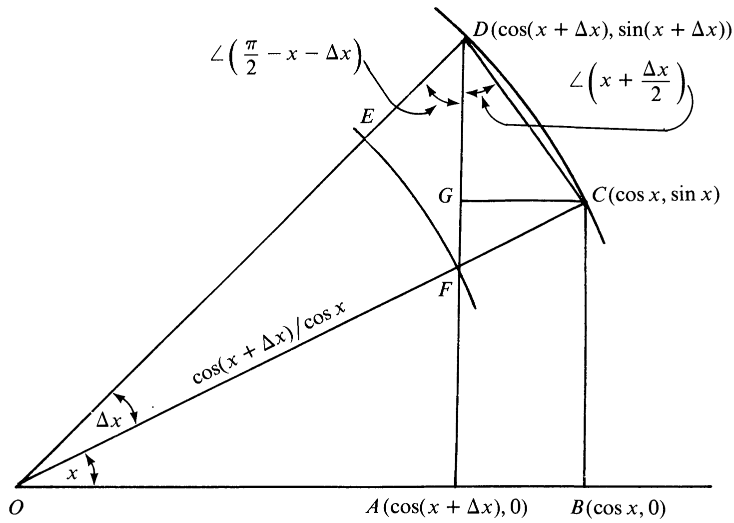

Figure 1.

In the figure, DC and EF are arcs of circles with center at O and radii 1 and $\cos(x + \Delta x)/\cos x$, respectively. It is easily verified that the length of chord DC is greater than the length of the inscribed arc tangent to DC at its midpoint (use the fact that $\tan \frac{\Delta x}{2} > \frac{\Delta x}{2}$), and the length of this arc is greater than arc EF. Thus,

$$\text{arc } DC > \text{chord } DC > \text{arc } EF. \qquad (*)$$

Since $DG = \sin(x + \Delta x) - \sin x$, it follows from right triangle DGC that chord $DC = \dfrac{\sin(x + \Delta x) - \sin x}{\cos\left(x + \frac{\Delta x}{2}\right)}$. Furthermore, arc $DC = \Delta x$ and arc $EF = (\cos(x + \Delta x)/\cos x) \cdot \Delta x$. Using $(*)$, we get

$$\Delta x > \frac{\sin(x + \Delta x) - \sin x}{\cos\left(x + \frac{\Delta x}{2}\right)} > \frac{\cos(x + \Delta x)}{\cos x} \cdot \Delta x$$

or

$$\cos\left(x + \frac{\Delta x}{2}\right) > \frac{\sin(x + \Delta x) - \sin x}{\Delta x} > \frac{\cos(x + \Delta x)}{\cos x} \cdot \cos\left(x + \frac{\Delta x}{2}\right).$$

Hence,

$$\frac{d}{dx}(\sin x) = \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} = \cos x.$$

The same figure can be used to show that $\frac{d}{dx}(\cos x) = -\sin x$. Using $GC = \cos x - \cos(x + \Delta x)$, it follows from right triangle DGC that chord $DC = \dfrac{\cos x - \cos(x + \Delta x)}{\sin\left(x + \frac{\Delta x}{2}\right)}$. Again, using (*), we get

$$\Delta x > \frac{\cos x - \cos(x + \Delta x)}{\sin\left(x + \frac{\Delta x}{2}\right)} > \frac{\cos(x + \Delta x)}{\cos x} \cdot \Delta x$$

or

$$-\sin\left(x + \frac{\Delta x}{2}\right) < \frac{\cos(x + \Delta x) - \cos x}{\Delta x} < \frac{\cos(x + \Delta x)}{\cos x}\left(-\sin\left(x + \frac{\Delta x}{2}\right)\right).$$

Thus,

$$\frac{d}{dx}(\cos x) = \lim_{\Delta x \to 0} \frac{\cos(x + \Delta x) - \cos x}{\Delta x} = -\sin x.$$

It should be remarked that, although the angles were restricted to the first quadrant, the chain rule can be used to readily extend these results to other quadrants. Finally, we observe that the formulas $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ are immediate consequences of our formulas, since

$$\lim_{h\to 0} \frac{\sin(x + h) - \sin x}{h} = \cos x \quad \text{and} \quad \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} = -\sin x$$

for $x = 0$.

Application of a Generalized Fibonacci Sequence

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In the November 1979 Classroom Capsules Column, Michael Chamberlain gave a solution to the following problem:

A fair coin is tossed repeatedly until n consecutive heads are obtained. What is the expected number of tosses e_n to conclude the experiment? (*)

This capsule offers a nice illustration of how a generalized Fibonacci sequence can be used to solve the above expectation problem.

Given the positive integer n in (*), let

$$f_i = \begin{cases} 0, & i = 1, 2, \ldots, n-1 \\ 1, & i = n \\ \sum_{k=1}^{n} f_{i-k}, & i > n. \end{cases} \tag{1}$$