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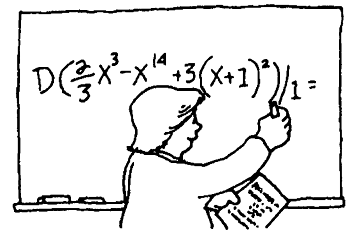

Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor, Warren Page, 30 Amberson Ave., Yonkers, NY 10705-3613.

The Logarithm Function and Riemann Sums

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In calculus integrals are usually introduced using Riemann sums with subintervals of equal length. In this note we approximate $\int_1^a 1/x\,dx$, $a > 0$, with Riemann sums with subintervals of unequal length to obtain the familiar limits

$$\lim_{n\to\infty} n(1 - a^{-1/n}) = \lim_{n\to\infty} n(a^{1/n} - 1) = \ln a.$$

The idea of using geometric progressions rather than arithmetic progressions is due to the seventeenth century French mathematician Pierre Fermat.

Assume that $a > 1$ (if $a < 1$, consider $1/a$) and form the geometric progression

$$1 = a^{0/n}, a^{1/n}, a^{2/n}, \dots, a^{n/n} = a.$$

These numbers, unequally spaced, partition the interval [1, a] into subintervals (see Figure 1). The first subinterval, $[1, a^{1/n}]$, is the smallest, with length $a^{1/n} - 1$. Since a partition of $[1, a]$ into n equal parts has subintervals of length $(a - 1)/n$, we see that

$$0 < a^{1/n} - 1 < \frac{a-1}{n}, \tag{1}$$

which implies the familiar result $\lim_{n\to\infty} a^{1/n} = 1$.

Now approximate $\int_1^a x\,dx$ with interior and exterior rectangles. Each interior rectangle has area

$$a^{-k/n}(a^{k/n} - a^{(k-1)/n}) = 1 - a^{-1/n}$$

and each exterior rectangle has area

$$a^{-(k-1)/n}(a^{k/n} - a^{(k-1)/n}) = a^{1/n} - 1.$$

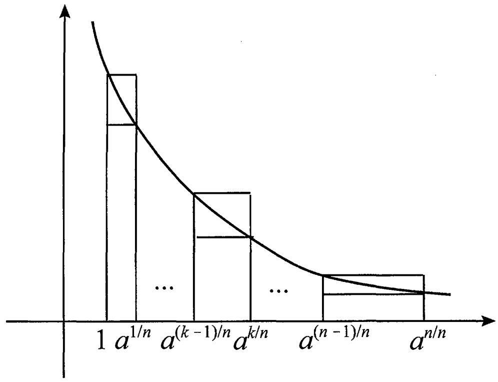

Figure 1.

Thus,

$$n(1-a^{-1/n}) < \int_1^a \frac{1}{x}\,dx < n(a^{1/n}-1).$$

Subtracting,

$$0 < \ln a - n(1-a^{-1/n}) < n(a^{1/n}-1) - n(1-a^{-1/n}) < n\left(\frac{a-1}{n}\right)^2,$$

where we used (1) and the assumption that $a > 1$ in the last inequality. Similarly,

$$0 < n(a^{1/n}-1) - \ln a < n\left(\frac{a-1}{n}\right)^2.$$

Taking limits, if $a > 0$,

$$\lim_{n\to\infty} n(1-a^{-1/n}) = \lim_{n\to\infty} n(a^{1/n}-1) = \ln a.$$

Exercise: Show that $\int_1^e \ln x\,dx = 1$ using Riemann sums.

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An Application of L'Hôpital's Rule

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Recently, while teaching a course in calculus, I asked my students to do the following exercise:

Let $f(x)$ be differentiable on an interval (a, ∞) and $\lim_{x\to\infty}(f'(x) + f(x)) = L$ (L may be infinite). Prove $\lim_{x\to\infty} f(x) = L$.

A solution given by a student raised my interest: let $g(x) = e^x f(x)$. Then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{g(x)}{e^x} = \lim_{x\to\infty} \frac{g'(x)}{e^x} = \lim_{x\to\infty} (f(x) + f'(x)) = L.$$

