Case 1. Say the barn wall forms one entire side of the pen as on the left in Figure 1. Let the pen have length y (parallel to the barn) and width x. Then $y + 2x = t$ and the area of the pen is $A = xy = x(t - 2x)$ for the domain $(t - b)/2 \leq x \leq t/2$. The lower bound on x follows from the assumption that $y \leq b$. Now $A'(x) = t - 4x$ so the maximal area occurs either at $x = t/4$, if this is in the domain, or else at the endpoint $x = (t - b)/2$. To summarize, the optimal pen using the barn wall as one entire side is described by

$$x = \frac{t}{4}, \, y = \frac{t}{2} \qquad \text{if } \frac{t - b}{2} \leq \frac{t}{4} \quad \text{(that is, } t \leq 2b\text{); or}$$
$$x = \frac{t - b}{2}, \, y = b \qquad \text{if } \frac{t}{4} \leq \frac{t - b}{2} \quad (2b \leq t).$$

Case 2. Say the barn wall forms only a part of one side of the pen, as on the right in Figure 1. Since $y \geq b$ the pen extends $y - b$ beyond the corner of the barn, and $y + 2x + (y - b) = t$. The area of the pen is now given by $A = xy = x(t + b - 2x)/2$ for the domain $0 \leq x \leq (t - b)/2$. The upper bound on x is derived from $y \geq b$. Since $A'(x) = (t + b)/2 - 2x$, the maximal area occurs either at $x = (t + b)/4$, if this is in the domain, or at the endpoint $x = (t - b)/2$. Thus the optimal pen when the barn is only a part of one side is described by
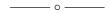
$$x = \frac{t - b}{2}, \, y = b \qquad \text{if } \frac{t - b}{2} \leq \frac{t + b}{4} \quad \text{(that is, } t \leq 3b\text{); or}$$
$$x = \frac{t + b}{4}, \, y = \frac{t + b}{4} \qquad \text{if } \frac{t + b}{4} \leq \frac{t - b}{2} \quad (3b \leq t).$$

All that remains is to compare the areas of optimal pens obtained by the two cases. But it is already apparent that if $2b \leq t \leq 3b$ then the two cases yield the same result: a pen with ratio $y/x = 2b/(t - b)$ whose maximal area occurs at an endpoint of the domain in each case. It is easy to see now that (1) correctly describes the ratio y/x of the dimensions of the optimal pen for each value of t.

The intermediate situation in which $2b < t < 3b$ is not often addressed in calculus texts. For example, if $t = \sqrt{5}b$, then the optimal pen forms a golden rectangle.

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Exploiting a Factorization of $x^n - y^n$

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Early in elementary algebra classes students factor $x^2 - y^2$, $x^3 - y^3, \ldots$, where x, y represent real numbers. Later they learn that

$$x^n - y^n = (x - y) \sum_{m=0}^{n-1} x^m y^{n-1-m} \tag{1}$$

for each positive integer n and all real x, y.

We noted a simple proof described by R. F. Johnsonbaugh [Another Proof of an Estimate for e, *American Mathematical Monthly* 81 (1974) 1011–1012], which shows that the sequence $\{(1 + 1/n)^n\}$ is increasing and bounded above, by appealing to this algebraic identity. First,

$$\left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n$$

$$= \left(1 + \frac{1}{n+1}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^{n+1} + \left(1 + \frac{1}{n}\right)^{n+1} - \left(1 + \frac{1}{n}\right)^n$$

$$= \left(\frac{1}{n+1} - \frac{1}{n}\right) \sum_{m=0}^{n} \left(1 + \frac{1}{n+1}\right)^m \left(1 + \frac{1}{n}\right)^{n-m} + \left(1 + \frac{1}{n}\right)^n \left[\left(1 + \frac{1}{n}\right) - 1\right]$$

$$> \left(-\frac{1}{n(n+1)}\right)(n+1)\left(1 + \frac{1}{n}\right)^n + \frac{1}{n}\left(1 + \frac{1}{n}\right)^n$$

$$= -\frac{1}{n}\left(1 + \frac{1}{n}\right)^n + \frac{1}{n}\left(1 + \frac{1}{n}\right)^n = 0.$$

So the sequence is increasing. To see that it is bounded above, note that by (1), for any integer $k > 1$,

$$\left(1 + \frac{1}{kn}\right)^n - 1 = \frac{1}{kn}\sum_{m=0}^{n-1}\left(1 + \frac{1}{kn}\right)^m < \frac{1}{kn}n\left(1 + \frac{1}{kn}\right)^n = \frac{1}{k}\left(1 + \frac{1}{kn}\right)^n.$$

Rearranging, $(1 + 1/kn)^n(1 - 1/k) < 1$, and raising both sides to the power k gives $(1 + 1/kn)^{kn}(1 - 1/k)^k < 1$, from which it follows that

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{kn}\right)^{kn} < \frac{1}{(1 - \frac{1}{k})^k}.$$

In particular, taking $k = 2$ gives $(1 + 1/n)^n < 4$ for all n.

We also found that combining (1) with the binomial theorem leads to a fundamental identity of binomial coefficients:

$$\sum_{k=1}^{n}\binom{n}{k}x^k = (x+1)^{n-1} - 1 = x\sum_{m=0}^{n-1}(x+1)^m = \sum_{m=0}^{n-1}\sum_{j=0}^{m}\binom{m}{j}x^{j+1}$$

$$= \sum_{k=1}^{n}\sum_{m=k-1}^{n-1}\binom{m}{k-1}x^k.$$

Then equating coefficients of x^k on both sides yields

$$\binom{n}{k} = \sum_{m=k-1}^{n-1}\binom{m}{k-1}.$$

We were motivated by the elegance and simplicity of the above proof that the sequence $\{(1 + 1/n)^n\}$ is increasing and bounded above, and the proof of the binomial coefficients identity, to investigate how (1) can be applied to produce simple proofs of other basic results. Could learning one identity early on prepare our students to establish a number of other known results and make other discoveries? Finding the following elegant proof, using (1), of the existence of nth roots further

encouraged us. This concrete example can help students make more sense of proofs of more general results, such as the intermediate value theorem.

Theorem. *If n is a positive integer and p is a real number such that $p^{n+1} \geq 0$, there is a unique real number x such that $x^n = p$ and $xp \geq 0$.*

Proof. Let $\mathbb{Q}$ be the set of rational numbers and let $S = \{r \in \mathbb{Q} : r \geq 0$ and $r^n \leq p^{n+1}\}$. Then $0 \in S$ and $1 + p^{n+1}$ is an upper bound for S. If s is the supremum of S, then for each positive integer k there is an $r_k \in S$ satisfying $s < r_k + 1/k$. It follows that the positive rational number $r_k + 1/k \notin S$, so these inequalities hold:

$$r_k^n \leq s^n < \left(r_k + \frac{1}{k}\right)^n \qquad \text{and} \qquad r_k^n \leq p^{n+1} < \left(r_k + \frac{1}{k}\right)^n.$$

Utilizing these inequalities and (1),

$$|s^n - p^{n+1}| \leq \left(r_k + \frac{1}{k}\right)^n - r_k^n = \frac{1}{k} \sum_{m=0}^{n-1} \left(r_k + \frac{1}{k}\right)^m r_k^{n-1-m}.$$

Now $(r_k + \frac{1}{k})^m \leq (s+1)^m$, since $0 \leq r_k \leq s$, and $r_k^{n-1-m} \leq (s+1)^{n-1-m}$, so $|s^n - p^{n+1}| \leq \frac{n}{k}(s+1)^{n-1}$. Since n and s are fixed but the integer k can be arbitrarily large, it follows that $s^n = p^{n+1}$. If $p = 0$, let $x = 0$; otherwise let $x = s/p$. Then x is the unique real number satisfying $x^n = p$ and $xp \geq 0$. $\quad\square$

Here are two more examples applying (1) to derive inequalities found in typical *analysis courses*.

Example 1. (Bernoulli's inequality) *If x is real and $1 + x \geq 0$, then $(1+x)^n \geq 1 + nx$ for each positive integer n.*

Proof. We see from (1) that for any integer $n > 1$

$$(1 + x)^n = 1 + x \sum_{m=0}^{n-1} (1 + x)^m$$

$$= 1 + x \left[1 + \sum_{m=1}^{n-1} \left(1 + x \sum_{k=0}^{m-1} (1 + x)^k \right) \right]$$

$$= 1 + nx + x^2 \sum_{m=1}^{n-1} \sum_{k=0}^{m-1} (1 + x)^k \geq 1 + nx.$$

This completes the proof. $\quad\square$

Example 2. *For any real numbers x and y, we can show that*

$$|x^n - y^n| \leq (|x - y| + |y|)^n - |y|^n. \tag{2}$$

Proof. By (1) and the triangle inequality,

$$|x^n - y^n| \le |x - y| \sum_{m=0}^{n-1} |x|^m |y|^{n-1-m}$$

$$\le |x - y| \sum_{m=0}^{n-1} (|x - y| + |y|)^m |y|^{n-1-m}.$$

By (1) again, this last expression equals $(|x - y| + |y|)^n - |y|^n$. $\square$

We close with some exercises.

Exercise 1. If n is a positive integer and (i) $f(x) = x^n$ or (ii) $f(x) = x^{1/n}$, derive the formula for $f'(a)$ using (1) and the definition

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Exercise 2. Use Bernoulli's inequality and (1) to prove that for $1 + x \ge 0$, $(1+x)^n \ge 1 + nx + (n/2)(n-1)x^2$.

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Divergence of the Harmonic Series by Rearrangement

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It is standard fare in calculus to prove the divergence of the harmonic series:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

One typically shows that the partial sums S_n grow without bound, generally by a "condensation" argument restricting n to the successive powers of 2. Alternatively or additionally, the integral test is called into play to compare the series to the integral

$$\int_1^\infty \frac{1}{x}\, dx.$$

Either approach affords the opportunity to look at the logarithmic growth of the partial sums, although the second one is more explicit.

I offer a quick proof by contradiction. It is not intended as a substitute for the above approaches, but students often benefit from seeing more than one. Moreover, it provides a nice application of the result that rearrangement of the terms in a convergent series of positive terms does not affect the sum. At the point in the course where the rearrangement theorem appears, we already know the basic facts needed now.

Proposition. The harmonic series diverges.