Colored Polygon Triangulations

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From the simple observation that each edge of a graph has two endpoints, it immediately follows that the sum of the valences of the vertices is twice the number of edges in the graph. Thus the sum of valences is even, and we quickly obtain the following result.

Handshaking lemma. *In any graph, there are an even number of vertices of odd valence.*

The reference to handshaking comes from the following application. Let the vertices of a graph represent people at a party, with an edge joining two vertices if and only if the corresponding two people have shaken hands at the party. Then an even number of people have shaken hands at the party with an odd number of people.

In this note I offer two more applications of the handshaking lemma. The first problem deals with coloring the sides of a polygon and any triangulation of the polygon. The complete solution is given. This problem works well as a classroom example or as assigned reading, and it is accompanied by several student exercises.

The second application replaces colorings of the sides with colorings of the vertices of a polygon and any of its triangulations. The vertex coloring problem is similar to the side coloring problem, but differs enough in its details to create a fresh problem. The solution of the vertex coloring problem is not given, leaving the problem to be used as a project topic in which students (perhaps collaborating in small groups) work out the details and write a report.

Coloring the sides of a triangulated polygon. Consider a polygon P whose sides are each colored red, blue, or green, according to the following rule.

Rule P. No adjacent sides of the polygon receive the same color.

An example of such a side-colored polygon is shown at the left in Figure 1. We shall refer to a vertex touching a red and a blue colored side as a red+blue vertex, with analogous designations for red+green and blue+green vertices.

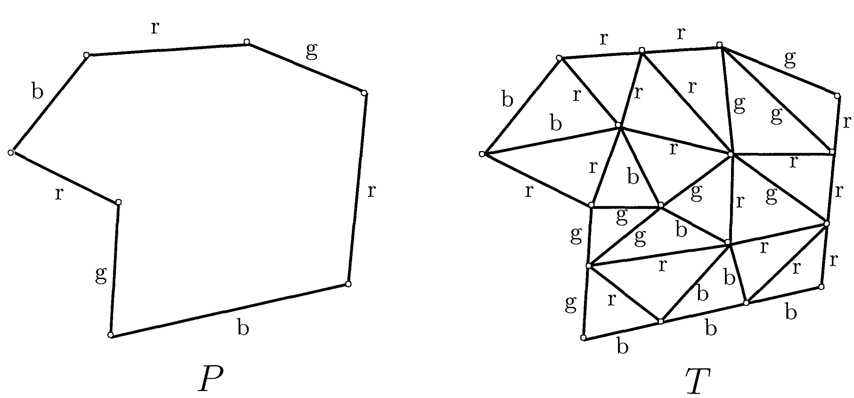

Figure 1. A side-colored polygon P and a side-colored triangulation T.

A triangulation T of polygon P appears at the right in Figure 1. Any two triangles of T either are disjoint or else intersect at a common vertex or along a common side. Vertices of T are allowed in both the interior of P and along the sides of P. The sides of T are colored red, blue, or green according to the following rule.

Rule T. Not all three colors appear on the sides incident to any vertex of T, and any side of T that is along an original side of P retains the color of that side.

We observe that rule T allows a vertex of T to be incident to sides all of one color. In Figure 1, an all-red vertex occurs along the uppermost red edge of T. Thus there are three one-color types of vertices of T in addition to the three two-color types of vertices already identified.

Within a side-colored triangulation T, must there exist a triangle that has one side of each color? We shall call such a triangle a *completely (side)-colored triangle*. If polygon P has evenly many sides, a valid side-coloring of P and any triangulation T can be given that uses just two colors. So in polygons with evenly many sides, the existence of a completely colored triangle cannot be guaranteed. The story is decidedly different when the polygon has an odd number of sides.

Problem 1. Let the sides of a polygon P be colored according to rule P, and let the sides of a triangulation T of P be colored according to rule T. If P has an odd number of sides, show that T contains at least one completely colored triangle.

Solution. Associate a graph G with T in the following way. Place one vertex in the interior of each triangle, and place one vertex in the exterior of P. An edge of G is introduced between two vertices precisely when the vertices belong to adjacent regions of T that have a common red side with one endpoint of type red+blue and the other endpoint of type all-red or red+green. The graph associated with the side-colored triangulation of Figure 1 is shown by the newly introduced vertices and edges shown in Figure 2. The graph G is easily drawn superimposed on T by identifying the red sides of T that have a red+blue type of endpoint and whose other

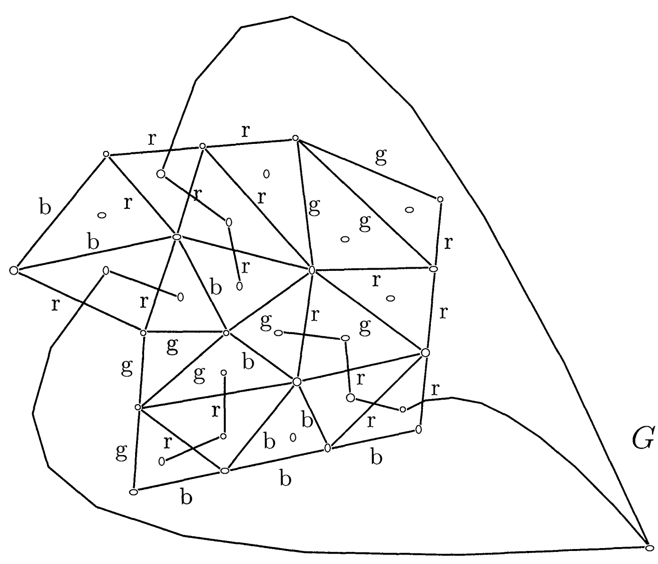

Figure 2. The graph G associated with the side-colored triangulation T.

endpoint is either red+green or all-red. Such sides are "crossed" by an edge of G that joins the vertices of G in the two regions bordering the red side.

The following two lemmas give information about the parity of the valence of any vertex in graph G.

Lemma 1. *The valence of any vertex of G inside a triangle of T is odd if and only if the triangle is completely colored, in which case the valence is* 1.

Proof. There are four cases to consider, depending on the number of red sides of the triangle corresponding to the vertex v of G.

1. *No side is red.* The valence of v is clearly 0.
2. *All three sides are red.* If either all or none of the vertices of the triangle touch blue sides of other triangles in the triangulation, the valence of v is 0. If either one or two vertices of the triangle touch blue sides then the valence of v is 2.
3. *Two sides are red.* If the third side of the triangle is green, the valence of v is either 2 or 0, depending on whether the common vertex of the red sides touches a blue side or not. Similarly, if the third side is blue, the valence of v is either 0 or 2.
4. *One side is red.* If the remaining two sides of the triangle are both blue or both green, the valence of v is 0. If the remaining sides are of different colors, namely one blue and one green, then the triangle is completely colored and v has valence 1. □

Lemma 2. *The valence of the vertex of G exterior to P has the parity of the number of vertices of P of type red+blue.*

Proof. First consider the case that no new vertices of T occur along any side of P. Any red side that is crossed by an edge of G has one red+blue endpoint and one red+green endpoint, so the valence of the exterior vertex of G is equal to the number of red+blue vertices that have a red+green vertex at the other endpoint of the red side. Any uncounted red+blue vertices occur in pairs, one at each endpoint of a red side, so these vertices do not affect the parity match. In the case that T has vertices on sides of the polygons, it is simple to check that such vertices add 0 or 2 to the valence and therefore leave the parity unchanged. □

Lemmas 1 and 2, together with the handshaking lemma, give us our main result.

Theorem. *Let the n sides of a polygon P be colored according to rule P, and let the sides of any triangulation T be colored according to rule T. Then the number of completely colored triangles, the number of red+blue vertices of the polygon, the number of blue+green vertices of the polygon, and the number of red+green vertices of the polygon all have the same parity as n.*

Proof. Applying the handshaking lemma to G, we conclude from lemma 1 that the number of completely colored triangles of T plus the valence of the vertex of G exterior to P is an even number. Thus the number of completely colored triangles has the same parity as the valence of this exterior vertex. Lemma 2 then implies that the number of completely colored triangles has the same parity as the number of vertices of the polygon of type red+blue. Now imagine all red, blue, and green edges of T are recolored blue, green, and red, respectively. The number of completely colored triangles is unchanged, so we conclude that in the original coloring of P the number of vertices of the polygon of type red+green also has the parity of the number of completely colored triangles. A similar argument applies to the blue+green vertices. Since n is the sum of three numbers of the same parity, n also has that parity. □

In the example shown in Figure 1, there are three vertices of type red+blue, one of type blue+green, and three of type red+green; $3 + 1 + 3 = 7 = n$ is the number of sides of the polygon. Since the smallest odd whole number is 1, we know that the triangulation must contain at least one completely colored triangle. In fact, there are five completely colored triangles in the example shown, each containing one of the valence 1 vertices of graph G.

Related exercises.

1. Let the sides of a polygon P be colored red, blue, or green. (Rule P is not enforced: consecutive sides of the polygon may be assigned the same color). Give a direct proof (not requiring a triangulation or the notion of completely colored triangles) that the numbers of vertices of each two-color type (red+blue, red+green, blue+green) have the same parity.

2. Let P be a polygon with sides colored according to rule P. Prove that any triangulation T of P can always be successfully colored following rule T.

3. Suppose that there are k red sides of P with endpoints of type red+blue and red+green.

a. Show that, for any colored triangulation T, the valence of the vertex of G exterior to P is at least k.

b. If $k = 3$, must T contain at least three completely colored triangles?

4. Characterize the structure of the graph G associated with a side-colored triangulation (describe cycles, components, and so forth).

5. Consider a triangulation T of a polygon P whose vertices are the n vertices of the polygon and i vertices in the interior of P (that is, no vertices of T are located along any side of P other than at endpoints). Let t and e denote the respective number of triangles and sides of T.

a. Prove that $3t + n = 2e$.

b. Prove that $t = 2i + n - 2$.

c. Prove that in a side-colored triangulation of T the parity of the number of triangles matches the parity of n.

Coloring the vertices of a triangulated polygon. Consider a polygon P whose vertices are each colored red, blue, or green, according to the following rule.

Rule P'. No adjacent vertices of the polygon receive the same color.

An example of such a vertex-colored polygon P is shown at the left in Figure 3. A triangulation T of the polygon appears at the right of the figure, where we see that new vertices are introduced in the interior of the polygon or along sides of the polygon. The new vertices of T are colored according to the following rule.

Rule T'. Any vertex of T that is on an original side of the polygon must be assigned either of the two colors at the endpoints of that side.

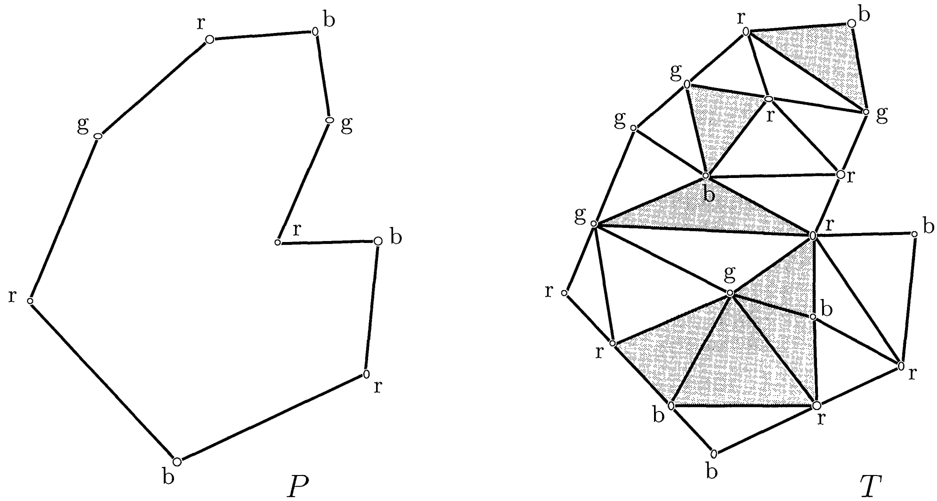

Figure 3. A vertex-colored polygon P and a vertex-colored triangulation T.

The sides of T with one red and one blue endpoint are said to be of type rb. Altogether, the sides of T are of six color types: rb, rg, bg, rr, bb, gg.

Figure 3 shows an example of a vertex-colored polygon P and a triangulation T. The 9-gon P has five sides of type rb, one of type bg, and three of type rg. The triangulation T on the right has seven triangles with vertices of all three colors. These *completely vertex-colored triangles* have been shaded.

Problem 2. Let the vertices of a polygon P be colored according to rule P', and let the vertices of a triangulation T of P be colored according to rule T'. If P has an odd number of sides, show that T contains at least one completely vertex-colored triangle.

Comments. The result of the side-coloring theorem, without proof, was published previously as one of several discovery activities [**3**]. For very different solutions to problem 1 in the triangle case $n = 3$, see [**1**] and [**2**]. Problem 2, restricted to the triangular case $n = 3$, is a generalization of Sperner's lemma, which is a key step in a proof of Brouwer's fixed point theorem. A graph-theoretic proof of Sperner's lemma can be found in [**4**].

References

1. Duane DeTemple and Jack Robertson, An analog of Sperner's lemma, *American Mathematical Monthly* 83 (1976) 465–467.
2. ———, Proofs by parity, *Mathematics Notes from Washington State University* 19:4 (November 1976).
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4. Alan Tucker, *Applied Combinatorics* 2nd ed., Wiley, New York, 1984, p. 27.

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