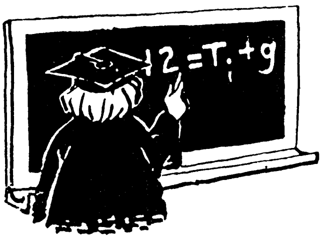

CLASSROOM CAPSULES

Edited by
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A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics.

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An Alternative to Changing the Order of Integration

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Students in calculus and analysis are often faced with iterated integrals of the form

$$\int_a^b \int_{\alpha(x)}^{\beta(x)} h(x,t)\,dt\,dx. \tag{1}$$

In some cases, the inner integral $\int_{\alpha(x)}^{\beta(x)} h(x,t)\,dt$ is difficult or impossible to evaluate, but we observe that there is a function $H(x,t)$ with $\partial H(x,t)/\partial x = h(x,t)$. With this observation, we change the order of integration and hope for the best.

Unfortunately, changing the order of integration often leads to complicated algebraic problems when it comes to finding the new limits of integration. Furthermore, if the region of integration is complicated, the revised integral may have to be expressed as the sum of several iterated integrals over smaller regions. The theorem presented below eliminates the algebraic difficulties mentioned above, and may result in having to evaluate fewer integrals than in the standard "reverse-of-order-of-integration" techniques. We shall assume that all functions considered are real-valued.

Theorem. *Let $\alpha(x)$ and $\beta(x)$ be continuously differentiable on $[a,b]$, and suppose $H(x,t)$ and $H_x(x,t) = h(x,t)$ are continuous on $D = \{(x,t): a \le x \le b,$*

$\alpha(x) \le t \le \beta(x)\}$. *Then*

$$\int_a^b \int_{\alpha(x)}^{\beta(x)} h(x,t)\,dt\,dx = \int_a^b [H(x,\alpha(x))\alpha'(x) - H(x,\beta(x))\beta'(x)]\,dx$$
$$+ \int_{\alpha(b)}^{\beta(b)} H(b,t)\,dt - \int_{\alpha(a)}^{\beta(a)} H(a,t)\,dt. \qquad (2)$$

Proof. By Leibniz's formula

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} H(x,t)\,dt = \int_{\alpha(x)}^{\beta(x)} h(x,t)\,dt + H(x,\beta(x))\beta'(x) - H(x,\alpha(x))\alpha'(x).$$

Therefore

$$\int_a^b \left(\int_{\alpha(x)}^{\beta(x)} h(x,t)\,dt \right) dx = \int_a^b \left(\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} H(x,t)\,dt \right) dx$$
$$+ \int_a^b [H(x,\alpha(x))\alpha'(x) - H(x,\beta(x))\beta'(x)]\,dx,$$

from which (2) immediately follows.

One remark is important to make at this point. If it is difficult or impossible to integrate $h(x,t)$ with respect to t, it is likely that the same may be true of $H(b,t)$ and $H(a,t)$. Thus, the last two integrals in (2) may present the same problem as the inner integral in (1). However, we shall show that if (1) is an iterated integral that can be evaluated by first changing the order of integration, then the integrals that arise in the theorem *can* be evaluated. The converse is also true. Thus, (2) can be successfully used precisely when changing the order of integration would also be successful.

To change the order of integration in (1), first partition the interval $[a, b]$ into subintervals on which $\alpha(x)$ and $\beta(x)$ are each strictly monotonic or constant. We then change the order of integration on each of these subregions $D_1, D_2, \ldots, D_n$ of D. (See Figure 1.)

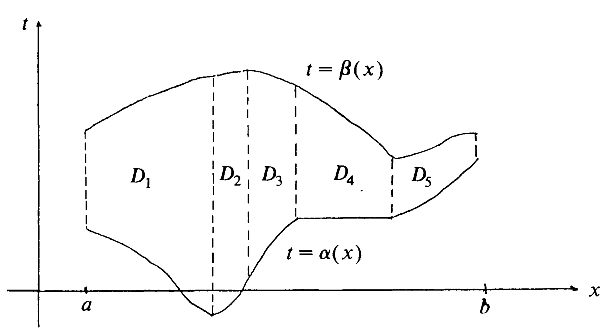

Figure 1.

To illustrate the proof of the above claim, consider the particular case in which $\alpha(x)$ is strictly increasing on $[a, b]$, and $\beta(x)$ is strictly decreasing on $[a, b]$ (see Figure 2). (The mechanics of this case are similar to those of other cases and would have to be carried out on each of the subregions formed in the previous discussion.)

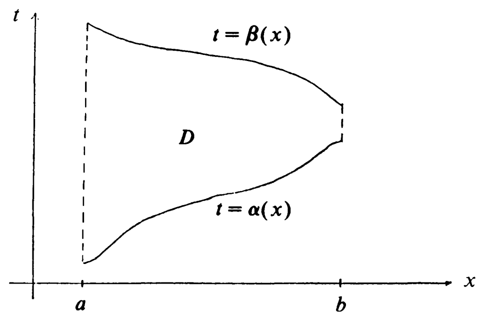

Figure 2.

We show that if the integral in (1) can be evaluated by changing the order of integration, then the integrals appearing on the right side of (2) can also be evaluated. Starting from the right-hand side of (2),

$$\int_a^b H(x, \alpha(x))\alpha'(x)\,dx - \int_a^b H(x, \beta(x))\beta'(x)\,dx$$
$$+ \int_{\alpha(b)}^{\beta(b)} H(b, t)\,dt - \int_{\alpha(a)}^{\beta(a)} H(a, t)\,dt, \tag{3}$$

make the substitution $t = \alpha(x)$ in the first integral, and $t = \beta(x)$ in the second. Then (3) becomes

$$\int_{\alpha(a)}^{\alpha(b)} H(\alpha^{-1}(t), t)\,dt - \int_{\beta(a)}^{\beta(b)} H(\beta^{-1}(t), t)\,dt + \int_{\alpha(b)}^{\beta(b)} H(b, t)\,dt - \int_{\alpha(a)}^{\beta(a)} H(a, t)\,dt. \tag{4}$$

For our case,

$$\alpha(a) \le \alpha(b) \le \beta(b) \le \beta(a) \tag{5}$$

which allows us to recast (4) as

$$\int_{\alpha(a)}^{\alpha(b)} [H(\alpha^{-1}(t), t) - H(a, t)]\,dt$$
$$+ \int_{\beta(b)}^{\beta(a)} [H(\beta^{-1}(t), t) - H(a, t)]\,dt$$
$$+ \int_{\alpha(b)}^{\beta(b)} [H(b, t) - H(a, t)]\,dt \tag{6}$$
$$= \int_{\alpha(a)}^{\alpha(b)} \int_a^{\alpha^{-1}(t)} h(x, t)\,dx\,dt + \int_{\alpha(b)}^{\beta(b)} \int_a^b h(x, t)\,dx\,dt + \int_{\beta(b)}^{\beta(a)} \int_a^{\beta^{-1}(t)} h(x, t)\,dx\,dt. \tag{7}$$

This last sum (7) is the integral in (1) over the region D in Figure 2 with the order of integration reversed. Since the above steps are reversible, we see that if the integrals in (2) can all be evaluated, then the double integral with the order of integration reversed can also be evaluated.

The following observations concerning $\alpha(x)$ and $\beta(x)$ should be made concerning the other cases. If α and β are strictly monotonic on $[a,b]$, then (4) always follows from (3). The natures of α and β become important in (5), where the relative sizes of $\alpha(a), \alpha(b), \beta(a), \beta(b)$ must be taken into account when "regrouping" the terms in (4). Variations of (6) consistent with the region D under consideration will then be obtained. If $\alpha(x)$ and/or $\beta(x)$ are constant on $[a,b]$, then $\alpha'(x) \equiv 0$ and/or $\beta'(x) \equiv 0$ on $[a,b]$. In this case, one or two integrals in (3) vanish and the calculations are simpler.

As the above argument shows, it is possible that in evaluating the integrals on the right side of (2), we might have to make the substitutions $x = \alpha^{-1}(t)$ and/or $x = \beta^{-1}(t)$ on appropriate subintervals of the x-axis. Of course we hope this is not necessary since the advantages of our theorem are that it allows us to avoid the computation of these inverse functions, and also to avoid having to partition $[a,b]$ into subintervals. In the many examples tried by the author, these substitutions and partitions have never been necessary. Thus, it appears that this theorem can be used to simplify calculations in otherwise difficult double integral evaluations.

We conclude with a general example, showing that in some cases it actually *is* easier to apply (2) than to reverse the order of integration. Let $u(x)$ be a differentiable function on $[a,b]$, where $0 < u(x) < 1$ on (a,b) and let $u(a), u(b) \in \{0,1\}$. Then consider

$$\int_a^b \int_{(u(x))^r}^{(u(x))^s} u^{n-1}(x)u'(x)\left[n + mt^k u(x)^m\right] e^{(u(x))^m t^k}\, dt\, dx, \tag{8}$$

where k, m, n, r, s are real numbers and $(n+r)/(m+rk)$ and $(n+s)/(m+sk)$ are both nonnegative integers.

The integral with respect to t cannot be evaluated, but we observe that the integrand is the derivative of

$$H(x,t) = (u(x))^n e^{(u(x))^m t^k}$$

with respect to x. Applying (2), we find (8) is equal to

$$\int_a^b r(u(x))^{n+r-1} e^{(u(x))^{m+rk}} u'(x)\, dx - \int_a^b s(u(x))^{n+s-1} e^{(u(x))^{m+sk}} u'(x)\, dx$$
$$+ \int_{(u(b))^r}^{(u(b))^s} (u(b))^n e^{(u(b))^m t^k}\, dt - \int_{(u(a))^r}^{(u(a))^s} (u(a))^n e^{(u(a))^m t^k}\, dt.$$

The last two integrals will be 0 since $u(a), u(b) \in \{0,1\}$ means each of these integrals has the same upper and lower limits. The first and second integrals can be evaluated by making the substitutions $w = (u(x))^{m+rk}$ and $y = (u(x))^{m+sk}$ respectively, then integrating by parts.

If $u(x)$ has several extrema on $[a,b]$, then although (8) can be evaluated by reversing the order of integration and keeping track of the interplay between $u(x)$ and $u^{-1}(x)$ on various intervals, the evaluation would be technically taxing to say the least.

Other examples similar to (8) may be constructed using trigonometric or logarithmic functions instead of exponential functions.