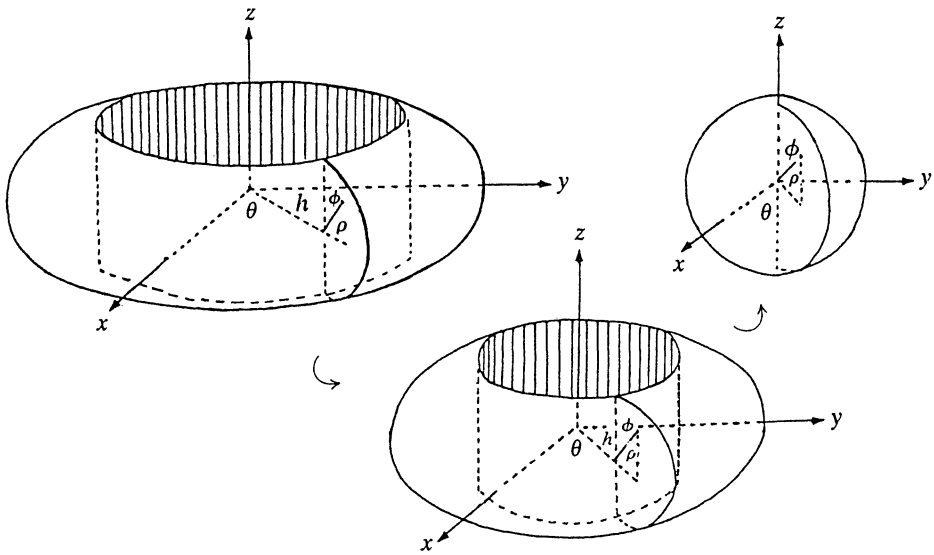

Figure 2

nate change (1) becomes

$$r = \rho \sin \phi, \qquad z = \rho \cos \phi, \qquad \theta = \zeta, \tag{1'}$$

and combining this with "ordinary" cylindrical coordinates $x = r \cos \theta, y = r \sin \theta$, $z = z$ gives the standard spherical coordinate transformation

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi.$$

Thus, the spherical coordinate transformation arises as a degenerate case of the double cylindrical coordinates on the torus. A natural question to ask at this point is "How are these coordinates related to the sphere?" Note that as $h \to 0$, the torus passes through itself and collapses to a ball of radius R. Each point in the ball is covered twice (except for the poles, which are covered infinitely many times). To remedy this duplication we restrict ϕ to range from 0 to π. It is easy to see that the resulting transform is one-to-one (except, again, at the poles). Figure 2 shows graphically how the coordinates (ρ, ϕ, θ) on the outer "half" of the torus become the standard spherical coordinates on the ball $\rho \le R$, as $h \to 0$.

MAD Property of Medians: An Induction Proof

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Recent elementary proofs of the widely known fact that any median for a sample of n numbers $x_1 \le x_2 \le \cdots \le x_n$ minimizes the mean absolute deviation (MAD) function

$$h_n(\alpha) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \alpha|$$

have appeared in **[1]** and **[2]**. Since this is a sequence of propositions, one for each positive integer, it seems natural to prove them using a mathematical induction argument. The proof may appeal to teachers as a good exercise for students who are learning to construct proofs by induction.

Proposition P_n. *Let $x_1 \leq x_2 \leq \cdots \leq x_n$ be any ordered set of n numbers. Then $h_n(\alpha)$ is minimized at any median of $x_1, \ldots, x_n$.*

Proof. Recall that the median of $x_1, \ldots, x_n$ is $x_{(n+1)/2}$ if n is odd, and if n is even then any number m with $x_{n/2} \leq m \leq x_{n/2+1}$ is a median. Let $\bar{x}$ be the sample mean, $(x_1 + x_2 + \cdots + x_n)/n$. Since $h_n(x_1) = \bar{x} - x_1 < \bar{x} - \alpha = h_n(\alpha)$ for $\alpha < x_1$, and since $h_n(x_n) = x_n - \bar{x} < \alpha - \bar{x} = h_n \alpha$ for $\alpha > x_n$, it suffices to consider $\alpha \in [x_1, x_n]$.

Observe that P_1 is trivially true. For $n = 2$ and $\alpha \in [x_1, x_2]$,

$$2h_2(\alpha) = (\alpha - x_1) + (x_2 - \alpha) = x_2 - x_1,$$

so P_2 is true.

Assume that P_i is true for all positive integers i, $1 \leq i \leq k$, for a fixed k, $k \geq 2$. Let $x_1 \leq x_2 \leq \cdots \leq x_{k+1}$ be any ordered set of $k+1$ numbers. Then for any $\alpha \in [x_1, x_{k+1}]$,

$$\begin{aligned}(k+1)h_{k+1}(\alpha) &= \sum_{i=1}^{k+1} |x_i - \alpha| \\ &= \alpha - x_1 + \sum_{i=2}^{k} |x_1 - \alpha| + x_{k+1} - \alpha \\ &= \sum_{i=2}^{k} |x_i - \alpha| + x_{k+1} - x_1.\end{aligned}$$

By the induction hypothesis, $\sum_{i=2}^{k} |x_i - \alpha|$, and hence h_{k+1}, is minimized at any median of the $k-1$ numbers $x_2, \ldots, x_k$. Since for $k \geq 2$ the set of medians of $x_1, \ldots, x_{k+1}$ is the same as the set of medians of $x_2, \ldots, x_k$, the validity of the proposition p_{k+1} follows. Thus, by the principle of mathematical induction, P_n is true for all natural numbers n.

The minimum value of $h_n(\alpha)$ is now easily found. Let $c = \lfloor (n+1)/2 \rfloor$, the greatest integer less than or equal to $(n+1)/2$. Then x_c is always a median of $x_1 \leq x_2 \leq \cdots \leq x_n$ and thus $\min_\alpha\{h_n(\alpha)\} = h_n(x_c)$, which is found to be

$$\frac{(x_n - x_1) + (x_{n-1} - x_2) + \cdots + (x_{n+1-c} - x_c)}{n}, \quad \text{where}$$

$$n + 1 - c = \begin{cases} c & \text{if } n \text{ is odd} \\ c+1 & \text{if } n \text{ is even.} \end{cases}$$

Thus, for example, the MAD of the data 2, 3, 3, 4, 6, 10 is

$$\frac{(10-2) + (6-3) + (4-3)}{6} = 2,$$

and this value is achieved by any α between 3 and 4. The graph of $h_6(\alpha)$ in this case is shown in Figure 1.

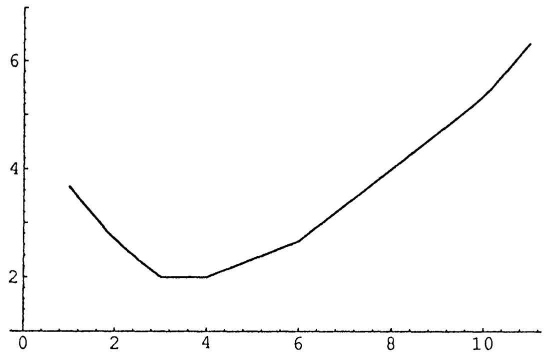

Figure 1

References

1. Kumar Joag-Dev, MAD property of a median: A simple proof, *American Statistician* 43 (1989) 26–27.
2. Neil C. Schwertman, A. J. Gilks, and J. Cameron, A simple noncalculus proof that the median minimizes the sum of absolute deviations, *American Statistician* 44 (1990) 38–39.

A Geometric Approach to Linear Functions

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Instead of drawing the traditional graphs, we will visualize linear functions as transformations of the real line $\mathscr{L}$. For example, the linear function $f(x) = \frac{4}{5}x + 1$ is illustrated by the 1-dimensional picture in Figure 1. In this picture, the coordinates of the points on $\mathscr{L}$ are listed below the line with the function values listed directly above. The action of this function on the line is indicated by the arrows: 0 is mapped to 1, so an arrow originates at 0 and extends to 1; 2 is mapped onto $\frac{13}{5}$, an increase of $\frac{3}{5}$, so the arrow from 2 has length $\frac{3}{5}$ and extends to the right; and so on. From this picture, we perceive a very nice geometric description for the action of f: It is the contraction by a factor of $\frac{4}{5}$ about the point 5 (the center of the contraction). Geometrically, it is clear that f is uniquely determined by its slope $\frac{4}{5}$ and its center 5. We will start our investigation by considering this geometric observation in algebraic terms. We will then use our algebraic results to study the geometry of the line, and finally we will use both our algebraic and our geometric results to consider linear difference equations.

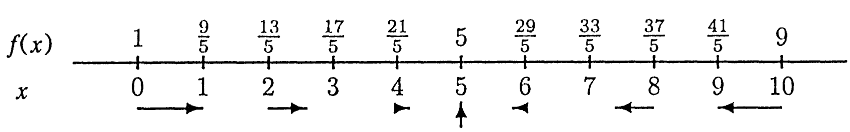

Figure 1

The slope-center form of a linear function. Many elementary courses start with a quick review of linear equations and their graphs. In these courses, we discuss the two-point form, the point-slope form and the slope-intercept form of the equation of a line. Our geometric approach motivates another useful form, the *slope-center* form. Consider the linear function $f(x) = ax + b$. Our first task is to discover if f has a center, or fixed point, and, if it does, to identify that point. Suppose x is a fixed point of f; then $x = ax + b$, or $(1 - a)x = b$. Clearly: