$$f'\left(\frac{a+x}{2}\right) = f'(d), \tag{7}$$

contradicting the fact that f' is strictly increasing on J. Our proof that $f'' = 0$ on I is thus complete.

Remarks. (a) For a proof of Theorem 3, emphasizing the continuity of f'', assume that $[a,x] \subset I$ and reason as above to obtain (7). Then, by the Mean Value theorem (or Rolle's theorem), $f''(x_1) = 0$ for some x_1 strictly between $(a+x)/2$ and x. Replace a in the preceding argument by x_1, thus obtaining an x_2 such that

$$\frac{\frac{a+x}{2}+x}{2} < \frac{x_1+x}{2} < x_2 < x$$

and $f''(x_2) = 0$. By iteration, we obtain a sequence $x_1 < x_2 < x_3 < \cdots$ such that all $f''(x_n) = 0$ and $\lim_{n\to\infty} x_n = x$. The continuity of f'' at $x \in I$ therefore yields $f''(x) = \lim_{n\to\infty} f''(x_n) = 0$.

(b) We state without proof the following companion for Theorem 3. *Let a real-valued function f be given on an open interval I containing* 0 *by a convergent power series* $\sum a_n x^n$. *Then f is linear if and only if I contains a positive number r such that* $M(f;\, 0,x) = f\left(\frac{x}{2}\right)$ *for all* $x \in (0,r)$.

On Rearrangements of the Alternating Harmonic Series

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The two series most familiar to beginning calculus students are the Harmonic Series (usually a student's first example of a divergent series whose terms approach zero) and the Alternating Harmonic Series (the first conditionally convergent series). When Taylor series are studied, it is shown that the Alternating Harmonic Series (abbreviated AHS) actually converges to $\ln 2$.

Because of its familiarity, the AHS is a reasonable candidate for illustrating how conditionally convergent series may be rearranged to change their sums. For example, we may replace each odd term x of the AHS by $(2x - x)$ and get a pattern in which one positive term is followed by two negative terms. If we then multiply each term of the new series by $1/2$, we get a rearrangement of the AHS which converges to half of the original sum. Thus,

$$\begin{aligned}\ln 2 &= 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots \\ &= 2 - 1 - \tfrac{1}{2} + \tfrac{2}{3} - \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{2}{5} - \tfrac{1}{5} - \tfrac{1}{6} + \cdots,\end{aligned}$$

and the rearranged AHS satisfies

$$\left(\tfrac{1}{2}\right)\ln 2 = 1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{5} - \tfrac{1}{10} - \cdots.$$

This is an example of a *regular rearrangement*, in which there is a regular pattern consisting of a fixed number of positive terms taken in order, followed by a fixed number of negative terms taken in order. We use $A(m,n)$ to denote such an ordered rearrangement consisting of *m positive terms* followed by *n negative terms*. Thus, the example above shows that $A(1,2)$ converges to $(1/2)\ln 2$. A simple argument [see Arthur B. Simon's *Calculus with Analytic Geometry*, Scott, Foresman and Co., Illinois (1982), 514] shows that $A(2,1)$ converges to $(3/2)\ln 2$.

Our investigation was initially prompted by a classroom problem posed by Edwin E. Moise [*Introductory Problem Courses in Analysis and Topology*, Springer-Verlag, New York (1982), 45]:

Find a rearrangement of the AHS that converges to zero.

The standard argument to show the existence of a rearrangement of the AHS which converges to a given limit L is to observe that because the Harmonic Series diverges, it is possible to add enough consecutive positive terms, $1 + 1/3 + \cdots$ to get a partial sum larger than L. Then enough consecutive negative terms are added to make the partial sum smaller than L, and the process is continued. For $L = 0$, a little work with a hand calculator showed some unanticipated regularity. To get a rearrangement of the AHS to converge to zero, it appeared that we needed *one* positive term and then *four* negative terms. The next positive term, $1/3$, made the sum positive, and four more negative terms were needed to get a negative partial sum. The same regularity continued as far as we were able to check by hand, so the class was led to search for a proof that this rearrangement $A(1,4)$ does, in fact, converge to zero. (A complicated inductive proof was discovered based on showing that the partial sums S_m are positive if 5 does not divide m, and S_m are negative if 5 divides m.)

Having discovered that the regular rearrangement $A(1,4)$ converges to zero, students began to ask questions about the convergence of $A(m,n)$ in general. We used a microcomputer to generate data on partial sums for a number of regular rearrangements. A few partial sums are given:

$A(1,1) \sim .6907$	$A(1,2) \sim .3453$
$A(2,2) \sim .6919$	$A(2,4) \sim .3459$
$A(3,3) \sim .6923$	$A(3,4) \sim .5489$
$A(4,4) \sim .6925$	$A(2,1) \sim 1.0372.$

Students were encouraged to formulate their own conjectures from the partial sums. For example, the data suggest that $A(2,2)$, $A(3,3)$ and $A(4,4)$ all converge to $\ln 2$. Similarly, it appears that $A(1,2)$ and $A(2,4)$ have the same limit. Ultimately we were led to the following:

Theorem. *Every regular rearrangement of the AHS converges. In particular,* $A(m,n)$ *converges to* $\ln 2 + (1/2)\ln(m/n)$.

Our proof of this theorem is based on the well-known fact [see, for example, Thomas and Finney's *Calculus and Analytic Geometry*, sixth edition, Addison-Wesley, Massachusetts (1984), p. 640, problem 45] that the difference $H_N - \ln N$ between H_N (the Nth partial sum of the Harmonic Series) and $\ln N$ approaches a constant γ (Euler's Constant) as N becomes large.

In working with partial sums for a rearrangement $A(m,n)$, it is most natural to consider sums of N terms, where N is divisible by $m + n$. For convenience of notation, O_N and E_N will, respectively, denote the sum of the first N odd terms and the sum of the first N even terms of the AHS. Thus, $O_N + E_N = H_{2N}$ and $2E_N = H_N$.

Let S_N be the Nth partial sum of $A(m,n)$, where $N = (m+n)k$. Collecting positive and negative terms together, we have:

$$S_{(m+n)k} = O_{mk} - E_{nk} = O_{mk} + E_{mk} - E_{mk} - E_{nk}$$

$$= H_{2mk} - \left(\frac{1}{2}\right)H_{mk} - \left(\frac{1}{2}\right)H_{nk}$$

$$= (H_{2mk} - \ln 2mk) - \left(\frac{1}{2}\right)(H_{mk} - \ln mk) - \left(\frac{1}{2}\right)(H_{nk} - \ln nk)$$

$$+ \ln 2mk - \left(\frac{1}{2}\right)\ln mk - \left(\frac{1}{2}\right)\ln nk$$

$$= (H_{2mk} - \ln 2mk) - \left(\frac{1}{2}\right)(H_{mk} - \ln mk) - \left(\frac{1}{2}\right)(H_{nk} - \ln nk)$$

$$+ \ln 2 + \left(\frac{1}{2}\right)\ln\left(\frac{m}{n}\right).$$

Therefore,

$$\lim_{k\to\infty} S_{(m+n)k} = \gamma - \left(\frac{1}{2}\right)\gamma - \left(\frac{1}{2}\right)\gamma + \ln 2 + \frac{1}{2}\ln\left(\frac{m}{n}\right)$$

$$= \ln 2 + \frac{1}{2}\ln\left(\frac{m}{n}\right).$$

For each fixed $r \in \{1,2,\ldots, m+n-1\}$, we have $S_{(m+n)k+r} = S_{(m+n)k} + \{r$ terms of $A(m,n)\}$. Since the terms of $A(m,n)$ approach zero, $\lim_{k\to\infty} S_{(m+n)k+r} = \ln 2 + (\frac{1}{2})\ln(m/n)$ for each r. Therefore, it is an easy matter to verify that $\lim_{n\to\infty} S_N = \ln 2 + (\frac{1}{2})\ln(m/n)$, even if N is not divisible by $m+n$.

There are some additional observations which may be pertinent here. As was pointed out earlier, there is a standard argument to show the existence of a rearrangement of the AHS which will converge to any given real number L. However, given an arbitrary real number L, there is not necessarily a *regular* rearrangement which will converge to L. From our theorem, we can prove the following:

Corollary. *There is a regular rearrangement of the AHS which converges to L if and only if e^{2L} is a rational number.*

If $A(m,n)$ converges to L, then $L = \ln 2 + (1/2)\ln(m/n)$ and $e^{2L} = 4m/n$ is rational. Suppose, conversely, that $e^{2L} = p/q$ is rational. Then $L = (1/2)\ln(p/q)$. If we choose integers m, n satisfying $m/n = p/4q$, then $A(m,n)$ converges to $\ln 2 + (1/2)\ln(p/4q) = L$.

As we had previously observed, there is not a unique rearrangement of the AHS that converges to a particular limit. More interestingly, we were led to consider regular rearrangements by following the standard argument for rearranging the AHS to converge to zero, but the standard rearrangement procedure does not necessarily lead to a regular rearrangement. Our Theorem shows that $A(4,1)$ converges to $\ln 4$; but if we had used the standard procedure to get a rearrangement converging to $\ln 4$, we would choose only *three* positive terms before the first negative term.

Editor's Note: For related discussions of this theme, see "Rearranging the Alternating Harmonic Series" by C. C. Cowen, K. R. Davidson, and R. P. Kaufman [Amer. Math. Monthly, 87 (December 1980) 817–819], "Sum-Preserving Rearrangements of Infinite Series" by Paul Schaefer [Amer. Math. Monthly, 88 (January 1981) 33–40], and "Rearranging Terms in Alternating Series" by Richard Beigel [Math. Mag., 54 (November 1981) 244–246].

Tetrahedra, Skew Lines, and Volume

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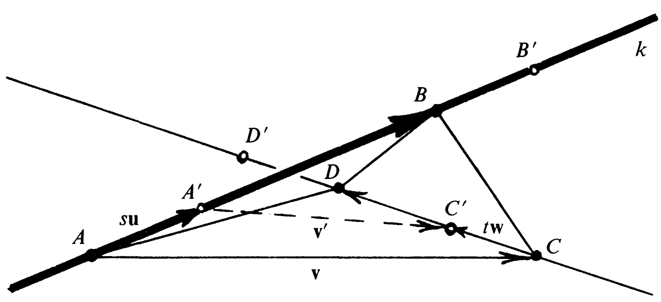
If A and B are distinct points on a line k, and C and D are distinct points on a line m that is skew to k, then A, B, C, and D determine a tetrahedron. This situation can be visualized by cutting a quadrilateral out of a sheet of paper and creasing it along a line between an appropriate pair of vertices. One of the two skew lines is associated with the crease and its two vertices, the other with the two remaining vertices.

Suppose, as above, that points, A, B, C, and D determine a tetrahedron for given fixed skew lines k and m. We intend to show that the volume of the tetrahedron does not change when segments AB and CD of fixed lengths are moved along their respective lines k and m.



To be specific, consider any points A', B' on k and points C', D' on m such that the lengths of segments $A'B'$ and $C'D'$ equal those of segments AB and CD, respectively. We consider the vector $\mathbf{u}'$ from A' to B' as being the same as the vector $\mathbf{u}$ from A to B, and the vector $\mathbf{w}'$ from C' to D' is considered the same vector as $\mathbf{w}$ from C to D.

Since A' lies on line k, the vector $\mathbf{AA}'$ from A to A' satisfies

$$\mathbf{AA}' = s\mathbf{u} \text{ from some scalar } s.$$