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Approximating Solutions for Exponential Equations

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To find x in $2^x = 5$, everyone will suggest, "use logarithms." But we are after something different. We would like to improve students' ability to manipulate exponents and increase their skills in estimation by obtaining x without the formal use of logarithms.

We begin by observing that $2^7 = 128$ is approximately equal to $5^3 = 125$. Thus, we write

$$5^3 \approx 2^7 \quad \text{or} \quad 5 \approx 2^{7/3} = 2^{2.333\ldots}$$

Since $2^{2.3219281\ldots} = 5$, our estimate is quite good (missing by only one-half of a percent.)

Next we seek relations that can be used to estimate 3, 7, and 11 each as powers of 2. We need only be concerned about prime integers since composites are products of primes. Thus, we want to approximate primes p as $p = 2^x$ for rational x.

For $p = 2$, clearly $x = 1$.

If $p = 3$, then $3^2 \approx 2^3$ yields

$$3 \approx 2^{3/2} \text{ (via logs, } 3 = 2^{1.5849625\ldots}\text{)}.$$

If $p = 7$, then $7^2 \approx 48 = 2^4 \cdot 3 \approx 2^4 \cdot 2^{3/2} = 2^{11/2}$ yields

$$7 \approx 2^{11/4} \text{ (via logs, } 7 = 2^{2.8073549}\text{)}.$$

If $p = 11$, then $11^2 \approx 120 = 2^3 \cdot 3 \cdot 5 \approx 2^3 \cdot 2^{3/2} \cdot 2^{7/3} = 2^{41/6}$ yields

$$11 \approx 2^{41/12} \text{ (via logs, } 11 = 2^{3.4594316\ldots}\text{)}.$$

In general, $p^2 \approx p^2 - 1 = (p-1)(p+1)$. Since $p-1$ and $p+1$ are composite, both factor into a product of primes all of which are less than p. Therefore, using the preceding approximations for each prime less than p, we obtain an approximation for p itself.

This algorithm may or may not produce better approximations for p than other generating relations. For example:

$$3^4 \approx 2^4 \cdot 5 \approx 2^4 \cdot 2^{7/3} = 2^{19/3} \text{ yields } 3 \approx 2^{19/12} = 2^{1.58333\ldots},$$

which is better than $3 \approx ((3+1)(3-1))^{1/2} = 2^{3/2}$.

$$5^2 \approx (5+1)(5-1) = 2^3 \cdot 3 \approx 2^3 \cdot 2^{19/12} = 2^{55/12} \text{ yields } 5 \approx 2^{55/24} = 2^{2.291666\ldots},$$

which is not as good as $5 \approx 2^{7/3}$.

$$7 \cdot 3^2 = 63 \approx 2^6 \text{ yields } 7 \approx 2^6/(2^{19/12})^2 = 2^{17/6} = 2^{2.8333\ldots},$$

which is better than $7 \approx 2^{11/4}$. These examples suggest that there is instructional value in seeking approximate relations that yield better approximations 2^x (x, rational) for the primes.

These techniques also lend themselves to compound interest problems, where equations such as $(1.12)^t = 1.4$ frequently arise. Since

$$1.12 = (7 \cdot 2^2)/5^2 \approx (2^{17/6} \cdot 2^2)/2^{14/3} = 2^{1/6}$$

and

$$1.4 = 7/5 \approx 2^{17/6}/2^{7/3} = 2^{1/2},$$

we find that $2^{t/6} \approx 2^{1/2}$ and $t \approx 3.0$. Since $t = 2.9689944\ldots$, via logs, our result is accurate to the nearest tenth.

A General Method of Deriving the Auxiliary Equation for Cauchy-Euler Equations

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The object of this note is to present a method for obtaining the auxiliary equation associated with the Cauchy-Euler linear differential equation of nth order

$$c_n x^n y^{(n)} + c_{n-1} x^{n-1} y^{(n-1)} + \cdots + c_1 x y^{(1)} + c_0 y^{(0)} = 0 \qquad (x > 0), \quad (\mathrm{CE})_n$$

where c_i $(0 \leqslant i \leqslant n)$ are constants with $c_n = 1$ and where

$$y^{(0)} = y(x) \quad \text{and} \quad y^{(i)} = \frac{d^i}{dx^i}\{y(x)\} \qquad \text{for} \quad i = 1, 2, \ldots, n.$$

For large values of n (specifically, for $n \geqslant 4$), the method described in textbooks to derive the auxiliary equation is time-consuming and laborious. We believe that the following approach is simple and elegant; it enables one to write the general solution of any Cauchy-Euler linear differential equation with considerable ease.

The usual method for deriving the auxiliary equation associated with $(\mathrm{CE})_n$ is to assume that $y(x) = x^m$ is a solution of $(\mathrm{CE})_n$. Then

$$y^{(i)} = m(m-1)(m-2)\cdots(m-i+1)x^{m-i},$$