CLASSROOM CAPSULES

EDITOR

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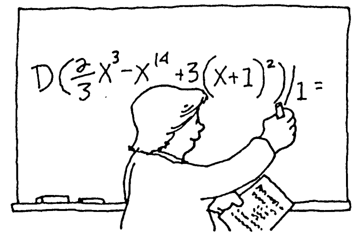

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Spherical Coordinates from Cylindrical Coordinates on a Torus

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During a lecture on triple integrals, I used cylindrical coordinates to compute the volume of a solid torus, i.e., a doughnut. While explaining the geometry to my students, I realized that spherical coordinates arise quite naturally from the double cylindrical coordinates on the torus. This point of view might make spherical coordinates more memorable; in any case, the discussion provides a good opportunity to emphasize the geometry behind changes of coordinates in space.

The region $T(R,h)$ in $\mathbb{R}^3$ obtained by revolving the disk $(y-h)^2+z^2 \leq R^2$ about the z-axis is the solid torus shown in Figure 1a (p. 386). This torus is described in cylindrical coordinates (r, θ, z) by the inequality $(r-h)^2+z^2 \leq R^2$. Thus the region in (r, θ, z)-space that corresponds to the solid torus is the right circular cylinder $C(R,h)$ of height 2π and base radius R, with central axis parallel to the θ-axis, at a distance h (see Figure 1b). Hence

$$\text{Volume}(T(R,h)) = \iiint_{T(R,h)} dx\,dy\,dz = \iiint_{C(R,h)} r\,dr\,d\theta\,dz.$$

Since the transformed integral in (r, θ, z)-space is a triple integral over a cylinder, it makes sense to change variables again using (slightly modified) cylindrical coordinates (ρ, ϕ, ζ):

$$r-h=\rho\sin\phi, \qquad z=\rho\cos\phi, \qquad \theta=\zeta. \tag{1}$$

Note that ρ measures the radial distance from the axis of the cylinder, and ϕ is the angle measured from the ray parallel to the positive z-axis through the axis of the cylinder at $\theta=\zeta$, as indicated in Figure 1b.

One readily computes that

$$\frac{\partial(r,\theta,z)}{\partial(\rho,\phi,\zeta)}=\rho,$$

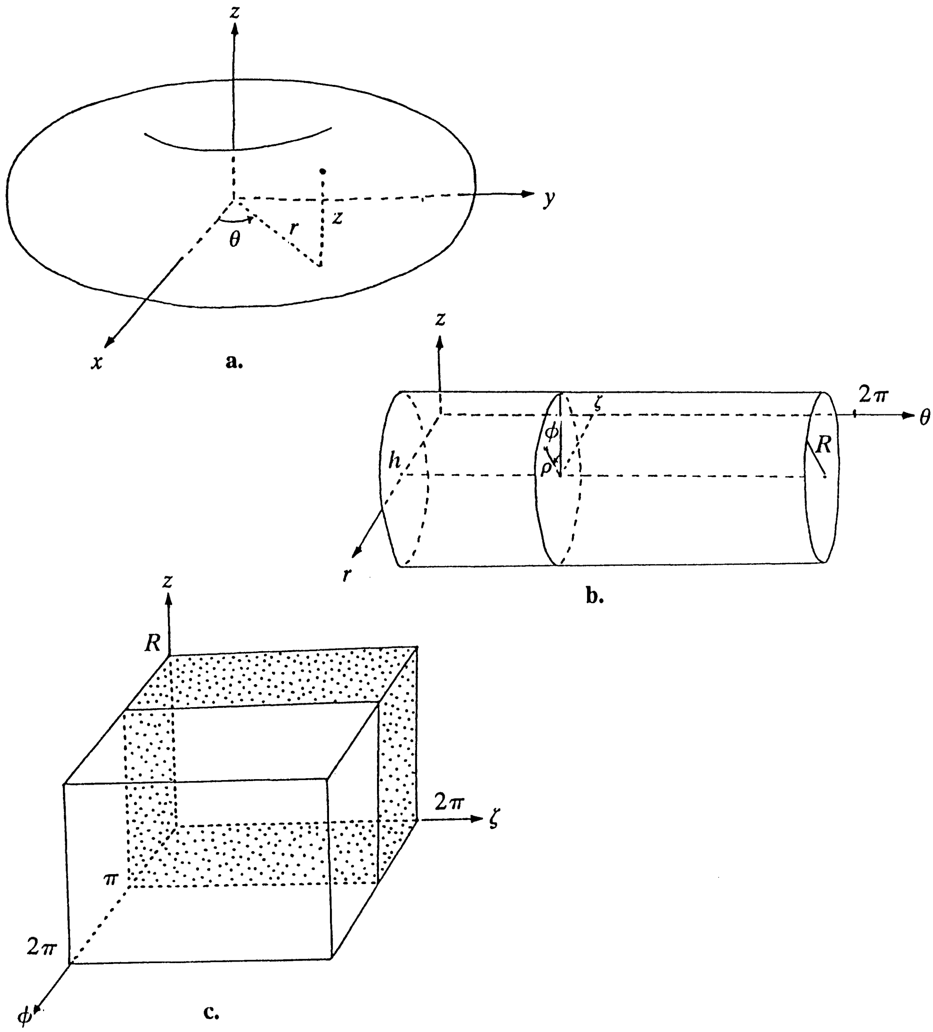

Figure 1

so

$$\iiint_{C(R,h)} r\,dr\,d\theta\,dz = \iiint_W (\rho\sin\phi + h)\rho\,d\rho\,d\phi\,d\zeta,$$

where W is the rectangular region $[0, R]\times[0,2\pi]\times[0,2\pi]$ in (ρ, ϕ, ζ)-space shown in Figure 1c. Thus

$$\begin{aligned}\iiint_W (\rho\sin\phi + h)\rho\,d\rho\,d\phi\,d\zeta &= \int_0^R\int_0^{2\pi}\int_0^{2\pi}(\rho\sin\phi + h)\rho\,d\zeta\,d\phi\,d\rho \\ &= 2\pi\int_0^R\int_0^{2\pi}(\rho\sin\phi + h)\rho\,d\phi\,d\rho = 2\pi^2R^2h.\end{aligned}$$

The alert reader will have spied in the triple integral over W the appearance of spherical coordinates in the form of the integrand $(\rho\sin\phi + h)\rho$. In fact, by letting $h = 0$ we see that the integrand becomes the Jacobian determinant $\rho^2\sin\phi$ for the transformation to spherical coordinates. Note that when $h = 0$ the coordi-

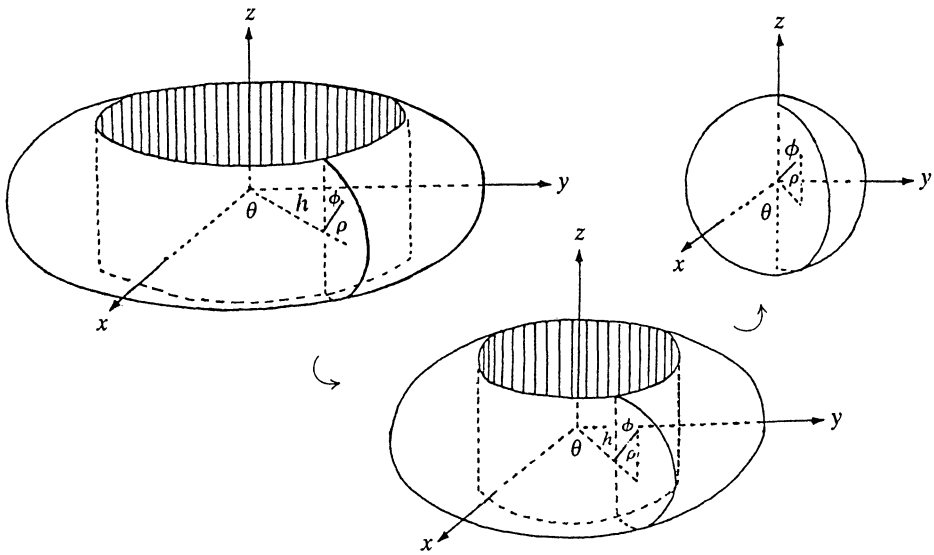

Figure 2

nate change (1) becomes

$$r = \rho \sin \phi, \qquad z = \rho \cos \phi, \qquad \theta = \zeta, \tag{1'}$$

and combining this with "ordinary" cylindrical coordinates $x = r \cos \theta, y = r \sin \theta$, $z = z$ gives the standard spherical coordinate transformation

$$x = \rho \sin \phi \cos \theta, \qquad y = \rho \sin \phi \sin \theta, \qquad z = \rho \cos \phi.$$

Thus, the spherical coordinate transformation arises as a degenerate case of the double cylindrical coordinates on the torus. A natural question to ask at this point is "How are these coordinates related to the sphere?" Note that as $h \to 0$, the torus passes through itself and collapses to a ball of radius R. Each point in the ball is covered twice (except for the poles, which are covered infinitely many times). To remedy this duplication we restrict ϕ to range from 0 to π. It is easy to see that the resulting transform is one-to-one (except, again, at the poles). Figure 2 shows graphically how the coordinates (ρ, ϕ, θ) on the outer "half" of the torus become the standard spherical coordinates on the ball $\rho \leq R$, as $h \to 0$.

MAD Property of Medians: An Induction Proof

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Recent elementary proofs of the widely known fact that any median for a sample of n numbers $x_1 \leq x_2 \leq \cdots \leq x_n$ minimizes the mean absolute deviation (MAD) function

$$h_n(\alpha) = \frac{1}{n} \sum_{i=1}^{n} |x_i - \alpha|$$

have appeared in [**1**] and [**2**]. Since this is a sequence of propositions, one for each positive integer, it seems natural to prove them using a mathematical induction