If $ABCD$ also has an incircle, then $K = \sqrt{abcd}$ [**11**]. We challenge the reader to find appropriate generalizations of the lemmas in this Capsule (and proofs, perhaps without words) for these expressions.

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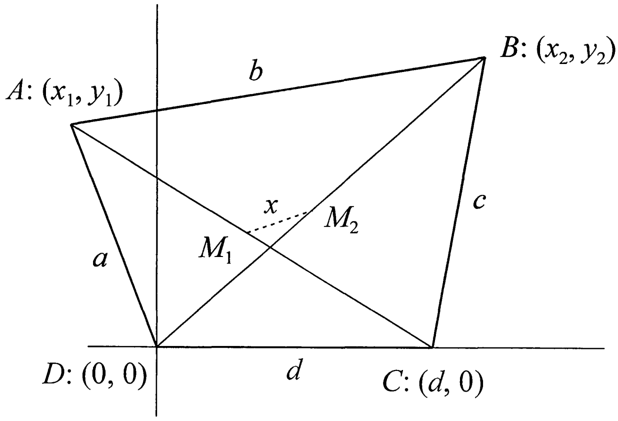
A Property of Quadrilaterals

Joesph B. Dence, University of Missouri, St. Louis, St. Louis, MO 63146 and Thomas P. Dence (tdence@ashland.edu), Ashland University, Ashland OH 44805

What if you were to draw a quadrilateral that was quite unlike a parallelogram? In an effort to disrupt the parallelism of pairs of opposite sides, you might construct the opposite interior angles to be as different as possible, or, equivalently, have the diagonals most clearly not bisect each other. The diagonals of a quadrilateral play a key role in many associated properties, especially with *cyclic* quadrilaterals, ones whose vertices lie on a circle [**1**]. For instance, Ptolemy's Theorem tells us that the product of the lengths of the diagonals of a cyclic quadrilateral equals the sum of the products of the lengths of the pairs of opposite sides. Cyclic quadrilaterals are also noteworthy because those are the ones of maximum area formed from four given sides. But even for an arbitrary quadrilateral, there is a remarkable relationship (originally proved by Euler) between the diagonals and the four sides. If we are given a convex quadrilateral $ABCD$, as shown in the figure, and a, b, c, d, are the lengths of the four sides, then the sum of the squares of these sides is related to the lengths of the two diagonals by

$$a^2 + b^2 + c^2 + d^2 = \overline{AC}^2 + \overline{BD}^2 + 4x^2,$$

where x is the length of the segment connecting the midpoints of the two diagonals. This $4x^2$ term can conveniently be thought of as measuring how far an ordinary quadrilateral differs from being a parallelogram.



In [**2**], Dunham gives Euler's proof of this result (using cyclic quadrilaterals) as well as a proof using the Law of Cosines. Here we present a simpler Law of Cosines proof.

One vertex D of the quadrilateral is conveniently placed at the origin, and side CD is placed along the x-axis, so that the coordinates of C are $(d, 0)$. Let the other two vertices be at $A : (x_1, y_1)$ and $B : (x_2, y_2)$. Applying the Law of Cosines to the four triangles ABD, CBD, BAC, and DAC gives

$$\overline{BD}^2 = a^2 + b^2 - 2ab\cos A = c^2 + d^2 - 2cd\cos C$$
$$\overline{AC}^2 = a^2 + d^2 - 2ad\cos D = b^2 + c^2 - 2bc\cos B$$

and it follows that

$$a^2 + b^2 + c^2 + d^2 = \overline{AC}^2 + \overline{BD}^2 + w,$$

where $w = ab\cos A + cd\cos C + ad\cos D + bc\cos B$. We then claim $w = 4x^2$.

To see this, first apply the distance formula to three of the sides of $ABCD$:

$$x_1^2 + y_1^2 = a^2$$
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = b^2$$
$$(d - x_2)^2 + (0 - y_2)^2 = c^2.$$

The four terms in w can now be written as

$$ab\cos A = \frac{a^2 + b^2 - \overline{BD}^2}{2} = \frac{x_1^2 + y_1^2 + (x_2 - x_1)^2 + (y_2 - y_1)^2 - x_2^2 - y_2^2}{2}$$
$$= x_1^2 + y_1^2 - x_1x_2 - y_1y_2$$
$$cd\cos C = \frac{c^2 + d^2 - \overline{BD}^2}{2} = \frac{(d - x_2)^2 + (0 - y_2)^2 + d^2 - x_2^2 - y_2^2}{2}$$
$$= d^2 - dx_2$$
$$ad\cos D = \frac{a^2 + d^2 - \overline{AC}^2}{2} = dx_1$$

$$bc\cos B = \frac{b^2 + c^2 - \overline{AC}^2}{2} = x_2^2 + y_2^2 - x_1x_2 - y_1y_2 - dx_2 + dx_1.$$

Summing these four equalities, with some minor rearrangements, gives

$$\begin{aligned} w &= ab\cos A + cd\cos C + ad\cos D + bc\cos B \\ &= \left[x_1^2 + 2dx_1 - 2x_1x_2 + d^2 - 2x_2d + x_2^2\right] + \left[y_1^2 + y_2^2 - 2y_1y_2\right] \\ &= 4\left[\left\{\frac{(x_1+d)}{2} - \frac{x_2}{2}\right\}^2 + \left\{\frac{y_1}{2} - \frac{y_2}{2}\right\}^2\right] \end{aligned}$$

and this last expression equals $4x^2$ since the midpoints have coordinates

$$M_1 = \left(\frac{x_1+d}{2}, \frac{y_1}{2}\right) \quad \text{and} \quad M_2 = \left(\frac{x_2}{2}, \frac{y_2}{2}\right).$$

We note in closing that if the vertices of the quadrilateral are located at $A:(0,2)$, $B:(2n+4, 2n+2)$, $C:(4,0)$, and $D:(0,0)$, then the distance between the two midpoints is $x = n\sqrt{2}$, and hence can be made as large as desired, reflecting the fact that a quadrilateral can be quite unlike a parallelogram.

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The Volume of a Tetrahedron

Cho Jinsok (jindol2@snu.ac.kr) student, Seoul National University

There are many formulas for calculating the area of a triangle, including one that is used when we know only the lengths of two sides and the angle between them. Likewise there are formulas for calculating the volume of a tetrahedron, but the common ones require the coordinates of the four vertices. Suppose, instead, we are given only the lengths of three edges having a common point and the measures of the three angles between these edges.

Theorem. For a tetrahedron $OABC$ let the angles $\angle AOB$, $\angle AOC$, and $\angle BOC$ have given values θ_1, θ_2, and θ_3, and let the lengths of the edges OA, OB, and OC be a, b, and c, respectively. Let $\theta = \frac{\theta_1+\theta_2+\theta_3}{2}$. Then the volume of the tetrahedron is given by

$$V = \frac{1}{3}abc\sqrt{\sin\theta\sin(\theta-\theta_1)\sin(\theta-\theta_2)\sin(\theta-\theta_3)}. \tag{2}$$

