A Simple Auxiliary Function for the Mean Value Theorem

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Most calculus texts contain the following statement:

Rolle's Theorem. *If f is continuous on $[a, b]$ and differentiable on (a, b) with*

$$f(a) = f(b) = 0, \tag{1}$$

then $f'(c) = 0$ for some $c \in (a, b)$.

Some texts replace (1) with

$$f(a) = f(b). \tag{1'}$$

This is certainly an improvement because an additional hypothesis is justifiable only if it delivers a stronger conclusion or an easier proof. I know of no proof of Rolle's theorem that makes use of the vanishing of f at the end points.

The main (only?) purpose of Rolle's theorem is to serve as a lemma for the more general mean value theorem. Whether (1) or (1′) is given in Rolle's theorem, the mean value theorem is usually proved by first introducing the auxiliary function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a) \tag{2}$$

and then demonstrating that it satisfies the hypotheses and hence the conclusion of Rolle's theorem. Despite our best efforts at making the function g in (2) appear geometrically intuitive and therefore "natural," most students seem to think of g as artificial and the proof as magical. It is our intention to show that the version of Rolle's theorem with (1) replaced by (1′) can lead to a more straightforward auxiliary function.

To establish the existence of a tangent line parallel to the secant line joining $(a, f(a))$ to $(b, f(b))$, it certainly makes sense to construct an auxiliary function that differs from f by a linear function whose slope is $(f(b) - f(a))/(b - a)$. But why not choose the simplest such linear function, the one passing through the origin? This is accomplished by replacing (2) with

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}x. \tag{2'}$$

The simplicity of the auxiliary function in (2′) more than offsets the computation needed to verify that it satisfies hypothesis (1′).

If instead we wish our auxiliary function g (satisfying (1′)) to agree with f at $x = a$ and still be a linear perturbation of f, then choose

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a).$$

An unnecessary translation downward by $f(a)$ brings us to $g(x) - f(a)$, the traditional auxiliary function given in (2).

We leave it for the reader to make a similar simplification in the proof of the generalized mean value theorem.