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Hyperbolic Functions and Proper Time in Relativity

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The hyperbolic functions have a connection with time dilation in relativity theory that can enrich discussions of these functions and the arc length formula in a calculus class.

It is well known that any point P on the unit circle $x^2+y^2=1$ has coordinates $(\cos s, \sin s)$, for a real number s that may be taken to be in $[0, 2\pi)$. The parameter s has several geometric interpretations (see Figure 1a):

(i) s is twice the area of the circular arc AOP, and
(ii) s is the length of the circular arc AP; that is, the radian measure of the central angle AOP.

For the *unit hyperbola* $x^2-y^2=1$, what analogous statements can be made?

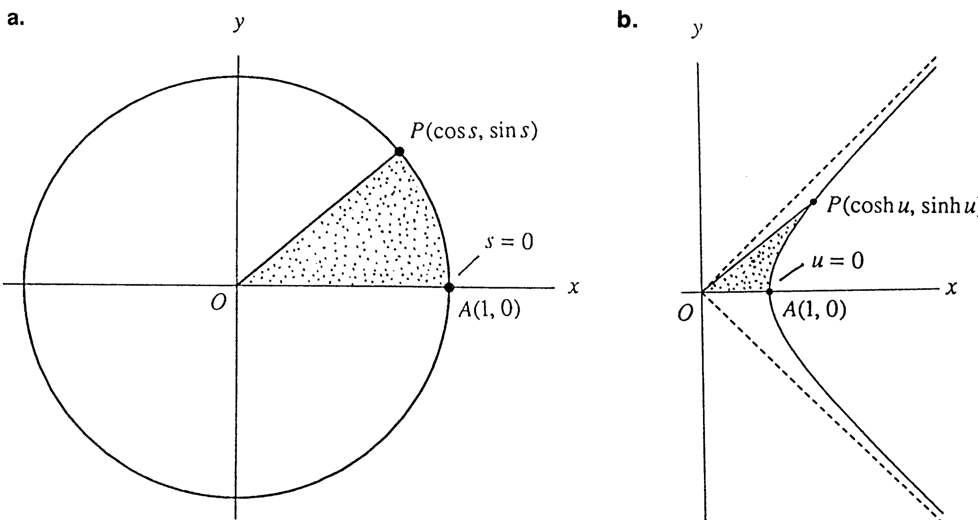

Figure 1

Any point P on the right branch of the unit hyperbola has coordinates $(\cosh u, \sinh u)$, as in Figure 1b, and it is well known [1] that in analogy with (i) above, u is twice the area of the hyperbolic sector AOP. Can u also be interpreted as an arc length, in analogy with (ii)?

The initial answer to this is "No"; an easy estimate shows that the length of the hyperbolic arc AP increases approximately exponentially with u. However, this is relative to the usual Pythagorean metric (infinitesimal arc length formula) $ds^2 = dx^2 + dy^2$ of Euclidean space. If instead the arc length is calculated relative to the Lorentz metric (sometimes called an *indefinite* metric or *pseudometric*) $du^2 = -dx^2 + dy^2$, then u is indeed the length of arc AP. When the usual arc length integral is modified to reflect the Lorentz metric, the Lorentz arc length of AP is

$$\int_0^u \sqrt{-x'(t)^2 + y'(t)^2}\, dt = \int_0^u \sqrt{-\sinh^2 t + \cosh^2 t}\, dt = u.$$

There is an interesting interpretation of the Lorentz arc length in special relativity theory. For simplicity, assume that "physical" space is one-dimensional (the x axis, calibrated in units of light-years). A vertical axis is introduced to show time in years and is labeled t rather than y. In these "normalized" units, the speed of light is 1. Two-dimensional *Minkowski spacetime* is the xt plane equipped with the Lorentz arc length described above.

If an observer is at location $x(t)$ at time t, his *worldline* is his trajectory in spacetime, the curve $(x(t), t)$. The time duration that he perceives between two events on his worldline is the Lorentz arc length of the segment of his worldline between these events. This is called the *proper time* between these events **[4]**, **[5]**. Thus the time duration experienced by the observer whose worldline is shown in Figure 2, between the two events indicated, would be $u = \int_a^b \sqrt{-x'(t)^2 + 1}\, dt$. This is an axiom of the model, and it is very well established empirically. Also well established experimentally is that the observer (or any object with nonzero rest mass) can never travel faster than light, or even quite that fast **[2]**. Thus $|x'(t)| < 1$, so the Lorentz arc length integrand is always real-valued.

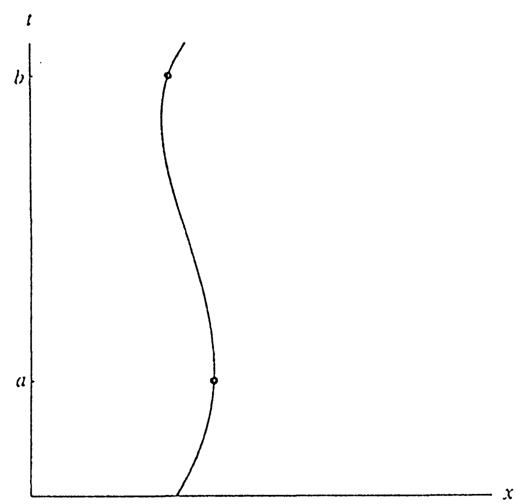

Figure 2

Uniformly accelerated motion. Suppose a rocket is launched at time $t = 0$ from the origin (Earth), and accelerates uniformly to the right along the x-axis, the constant acceleration being 1 light-year/year^2 (about the same as the acceleration due to gravity on Earth, quite comfortable for the crew). The worldline of the rocket, for $t > 0$, will be shown below to be the hyperbola $x(u) = \cosh u - 1, t(u) = \sinh u$, where the parameter u is the proper time perceived by the astronauts. Thus, when they felt that u years had passed aboard the rocket, their location would be at $x = \cosh u - 1$, while back on Earth (at $x = 0$), $t = \sinh u$ years would have passed. If the ship accelerated at one gravity, say for 20 years, then several hundred million irreversible years would have passed on earth. The rapid growth of t with u vividly illustrates the terrible loneliness of relativistic travel that is a frequent theme in science fiction classics [**3**].

Note that the accelerating rocket can "outrun" a light beam (and thus anything else), given a head start of one light-year or more. The worldline of a light beam in the positive direction is a line of slope 1, an asymptote of the rocket's hyperbolic worldline (Figure 1b). Thus the beam never quite overtakes the rocket.

Here is a sketch of the proof that the worldline of an object starting from rest and accelerating uniformly is a hyperbola, a well-known result in special relativity theory [**4**], [**5**]. Suppose the object, which we continue to refer to as a rocket, accelerates at a constant g light-years/year^2. Let the rocket's worldline be parametrized by the proper time u, say $x = x(u)$, $t = t(u)$, so the 4-*velocity* $V = (v_x, v_t)$ is $(x'(u), t'(u))$ and the rocket's 4-*acceleration* $A = (a_x, a_t)$ is $(x''(u), t''(u))$.

Using the definition of proper time we obtain $-x'(u)^2 + t'(u)^2 = 1$, or

$$v_x^2 - v_t^2 = -1. \tag{1}$$

Differentiating,

$$v_x a_x - v_t a_t = 0, \tag{2}$$

that is, the velocity and acceleration vectors of the rocket are Lorentz orthogonal at each instant [**6**].

It can be shown that in the instantaneous inertial rest frame of the rocket $a_x = g$ and $a_t = 0$, so $a_x^2 - a_t^2 = g^2$. But the expression $a_x^2 - a_t^2$ (the magnitude of the 4-vector A) is invariant, independent of the inertial frame, so the equation

$$a_x^2 - a_t^2 = g^2 \tag{3}$$

remains true in the standard xt coordinate system of the Earth in which our other calculations take place. Starting with (3), substituting the expression for a_t, from (2) and using (1) to eliminate v_x, we obtain

$$a_x = g v_t, \qquad \text{or} \qquad v_x'(u) = g v_t(u). \tag{4}$$

Again using (2) yields

$$a_t = g v_x, \qquad \text{or} \qquad v_t'(u) = g v_x(u). \tag{5}$$

When $u = 0$, at the start of the journey, $v_x(0) = 0$; and from (1), $v_t(0) = 1$. The system of first-order differential equations (4), (5) with those initial conditions has the unique solution

$$v_x = \sinh gu$$
$$v_t = \cosh gu$$

Integrating, using the initial conditions $x(0) = x_0$, $t(0) = 0$, then leads to the desired parametric equations for the rocket's trajectory in spacetime:

$$\begin{aligned} x(u) &= \frac{1}{g}(\cosh gu - 1) + x_0 \\ t(u) &= \frac{1}{g}\sinh gu. \end{aligned} \tag{6}$$

Note that the acceleration due to gravity on Earth, approximately 32 ft/s^2, works out to be about 1.03 light-years/year^2, so a constant acceleration of $g = 1$ is just under one Earth gravity. Also note that the second-order Taylor approximations to (6) yield precisely the nonrelativistic results: for small values of u

$$x(u) \approx \tfrac{1}{2}gu^2 + x_0$$
$$t(u) \approx u.$$

Dramatizing the parameter in the hyperbolic functions as the time experienced by a uniformly accelerating space traveler adds human interest to the study of these functions. Indeed, my students enjoy using the equations (6) to check the accuracy of their favorite science fiction writers.

Acknowledgment. I thank Glenn Calkins for a valuable discussion of these matters.

References

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For the Record

There is a typographical error on page 288 of Paulo Ribenboim's article, "Prime Number Records" [*CMJ* 25:4 (1994) 280–290]. An alert reader, Joe F. Wampler, noticed that the value given there for $\pi(10^{17})$ differs from that in Silviu Guiasu's recent article, "Is There Any Regularity in the Distribution of Prime Numbers at the Beginning of the Sequence of Positive Integers?" [*Mathematics Magazine* 68:2 (1995), Table IV on p. 120]. He inquired of Professor Guiasu, who observed that "Such a value of $\pi(10^{17})$ would make Riemann turn over in his grave, because his approximation would have an error greater than 2×10^{12} instead of only 598,254."

The correct value is $\pi(10^{17}) = 2{,}623{,}557{,}157{,}654{,}232$.