A particular case in point is $k=3$ and $j=5$. Here we easily solve (11),

$$(n+5)(n+4)(n+3)=\alpha_0+\alpha_1 n+\alpha_2 n(n-1)+\alpha_3 n(n-1)(n-2),$$

and obtain

$$D^5(x^3e^x)=(60+60x+15x^2+x^3)e^x.$$

For further details and related applications of (3), see Richard Brualdi's *Introductory Combinatorics*, North Holland, New York, 1983; E. S. Page and L. B. Wilson's *An Introduction to Computational Combinatorics*, Cambridge Computer Science Texts 9, 1979; and John Riordan's *An Introduction to Combinatorial Analysis*, John Wiley & Sons, New York, 1958.

Generalizations of a Complex Number Identity

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A recurring exercise that appears in texts on complex variables is to show that if w and z are complex numbers, then

$$|w|+|z|=\left|(w+z)/2-\sqrt{wz}\right|+\left|(w+z)/2+\sqrt{wz}\right|.$$

In problem 368, this journal, the first author asked for a generalization to three complex numbers. In this note, we give further generalizations to any number of variables and to any dimensional Euclidean space by replacing the complex numbers by vectors.

First, we can simplify the identity by getting rid of the bothersome square roots. Letting $w=z_1^2$ and $z=z_2^2$, we get

$$2\{|z_1|^2+|z_2|^2\}=|z_1-z_2|^2+|z_1+z_2|^2. \tag{1}$$

Geometrically, we now have that the sums of the squares of the edges of a parallelogram equals the sum of the squares of the diagonals. Consequently, by considering a parallelepiped, one generalization is that

$$\begin{aligned}4\{|z_1|^2+|z_2|^2+|z_3|^2\}&=|z_1+z_2+z_3|^2+|z_1+z_2-z_3|^2\\&\quad+|z_1-z_2+z_3|^2+|-z_1+z_2+z_3|^2.\end{aligned} \tag{2}$$

Here, z_1, z_2, z_3 can be complex numbers in the plane or vectors in space. For a proof, assuming the z_i are vectors, just note that

$$\begin{aligned}|z_1+z_2-z_3|^2&=(z_1+z_2-z_3)^2\\&=z_1^2+z_2^2+z_3^2+2z_1\cdot z_2-2z_1\cdot z_3-2z_2\cdot z_3,\ \text{etc.}\end{aligned}$$

Geometrically, we have that the sums of the squares of all the edges of a parallelepiped equals the sums of the squares of the four body diagonals. Also to be noted is that (1) is the special case of (2) when $z_3=0$. A generalization to n-dimensional space (for an n-dimensional parallelepiped) is immediate, i.e.,

$$2^n\sum z_i^2=\sum(\pm z_1\pm z_2\pm\cdots\pm z_n)^2, \tag{3}$$

where the summation on the right is taken over all the 2^n combinations of the $\pm$ signs.

For a generalization in another direction, note that (1) can be rewritten as

$$|z_1|^2+|z_2|^2=\left|(z_1-z_2)/\sqrt{2}\right|^2+\left|(z_1+z_2)/\sqrt{2}\right|^2.$$

In the real plane, the transformation $x'=(x-y)/\sqrt{2}$, $y'=(x+y)/\sqrt{2}$, represents a rotation of the coordinate axes by $45°$ and preserves all distances, i.e., $\sqrt{x^2+y^2}=\sqrt{x'^2+y'^2}$. For the case here, the value of $|z_1|^2+|z_2|^2$ is preserved under an orthogonal transformation. More generally (as is known), if $z_1, z_2, \dots, z_n$ are complex numbers (or vectors in space) and we make the transformation $Z'=MZ$ where M is an arbitrary real orthogonal matrix and the transpose matrices of Z and Z' are

$$Z^T=(z_1, z_2, \dots, z_n) \quad \text{and} \quad Z'^T=(z_1', z_2', \dots, z_n'),$$

then

$$\sum|z_i'|^2=\sum|z_i|^2$$

and its proof is quite direct:

$$\sum|z_i'|^2=\sum z_i'\bar{z}_i'=Z'^T\bar{Z}'=(MZ)^T(\overline{MZ})=Z^TM^TM\bar{Z}=Z^T\bar{Z}=\sum|z_i|^2$$

(since M is orthogonal $M^TM=I$). The proof for vectors is the same except that the multiplication of the two vector matrices Z^T and $\bar{Z}$ is via the scalar dot product.

More generally, the matrix M can be replaced by a complex matrix U if it is unitary, i.e., $U^T\bar{U}=I$. Finally, the identity (3) can be generalized by replacing the z_i's by z_i''s and then letting $Z'=UZ$. For a simple example in (2), let

$$z_1'=iz_1\cos\theta+iz_2\sin\theta, \qquad z_2'=z_1\sin\theta-z_2\cos\theta, \qquad z_3'=z_3$$

where θ is an arbitrary real angle.

A Generalization of $\lim_{n\to\infty}\sqrt[n]{n!}/n=e^{-1}$

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In "Alternate Approaches to Two Familiar Results" [CMJ 15 (November 1984) 422–423], we gave an elementary proof of the familiar result

$$\lim_{n\to\infty}\frac{(n!)^{1/n}}{n}=e^{-1}. \tag{1}$$

The following generalization of (1) states that for any nonnegative integer s:

$$\lim_{n\to\infty}\frac{\left(1^{(1^s)}\cdot 2^{(2^s)}\cdots n^{(n^s)}\right)^{1/n^{s+1}}}{n^{1/(s+1)}}=e^{-1/(s+1)^2}. \tag{2}$$