$n = 2$, the solution is $x = y = u - 1 = z - 2$. The generalization of equations (1) and (2) can continue, but as Fermat so aptly put it, the margin is too small.

Other cases of Fermat's craniosis have recently come to public attention. It is alarming that a preponderance of these documented cases have occurred within the mathematical community. In some cases, the infection has been traced back 400 years to an association with Fermat himself. The disease is eventually fatal, and the most debilitating effect is the patient's zealous desire to study point-set topology.

The good news is that only number theorists are susceptible to infection.

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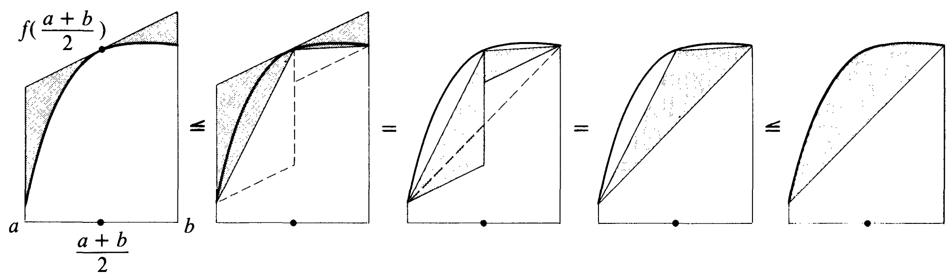
Behold! The Midpoint Rule is Better Than the Trapezoidal Rule for Concave Functions

Frank Buck, California State University, Chico, CA



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The Bisection Algorithm is Not Linearly Convergent

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The bisection algorithm is a method for finding a root of the equation $f(x) = 0$, where f is continuous on $[a,b]$ and $f(a)f(b) < 0$. Let $a(1) = a$, $b(1) = b$ and let $p(1)$ be the midpoint of $[a(1), b(1)]$. Now let $[a(2), b(2)]$ be one of the halves of this interval for which $f[a(1)]f[b(2)] < 0$, and let $p(2)$ be the midpoint of $[a(2), b(2)]$. Proceeding in this manner, the bisection algorithm generates a sequence of bracketing intervals $[a(i), b(i)]$ and a sequence of midpoints $p(i)$ of these intervals. It is well known that $p(i)$ converges to a root p of f. Some authors go even further and assert that $p(i)$ converges to p linearly—that is, there is some integer N and some constant Q such that $|p(i+1) - p| \leqslant Q|p(i) - p|$ for all $i \geqslant N$. This is not true in general. Although other authors have not made such an assertion, it is surprising that they give no counterexample. Here is a counterexample.

Let $p = 2^{-1} + 2^{-4} + \cdots + 2^{-n^2} + \cdots$ and let $f(x) = x - p$. Let the first braketing interval of the root p be $[2^{-1}, 1]$. Then it is easily verified that the second and the third bracketing intervals are $[2^{-1}, 2^{-1} + 2^{-2}]$ and $[2^{-1}, 2^{-1} + 2^{-3}]$. In general, let $S(n) = \{n^2, n^2 + 1, \ldots, (n+1)^2 - 1\}$ for $n = 1, 2, \ldots$. Then these sets form a partition of the set of positive integers. We assert that for $i \in S(n)$, the ith bracketing interval is

$$\left[a(i), b(i)\right] = \left[2^{-1} + 2^{-4} + \cdots + 2^{-n^2}, 2^{-1} + 2^{-4} + \cdots + 2^{-n^2} + 2^{-i}\right]. \quad (*)$$

The proof is by induction. We have seen above that $(*)$ holds for each $i \in S(1)$.

Assume that (*) holds for each $i \in S(n)$. Then

$$p\big((n+1)^2-1\big) = \Big[a\big((n+1)^2-1\big) + b\big((n+1)^2-1\big)\Big]/2$$

$$= 2^{-1} + 2^{-4} + \cdots + 2^{-n^2} + 2^{-(n+1)^2},$$

which is strictly less than p. Accordingly,

$$a\big((n+1)^2\big) = p\big((n+1)^2-1\big) = 2^{-1} + 2^{-4} + \cdots + 2^{-(n+1)^2}$$

and

$$b\big((n+1)^2\big) = b\big((n+1)^2-1\big) = 2^{-1} + 2^{-4} + \cdots + 2^{-(n+1)^2+1}$$

$$= 2^{-1} + \cdots + 2^{-(n+1)^2} + 2^{-(n+1)^2}.$$

This shows that $[a(i), b(i)]$ satisfies (*) for $i = (n+1)^2$ in $S(n+1)$. Suppose $[a(i), b(i)]$ satisfies (*) for $i = (n+1)^2 + k$ $(0 < k < 2n+2)$ in $S(n+1)$. Then

$$p\big((n+1)^2+k\big) = \Big[a\big((n+1)^2+k\big) + b\big((n+1)^2+k\big)\Big]/2$$

$$= 2^{-1} + 2^{-4} + \cdots + 2^{-(n+1)^2} + 2^{-[(n+1)^2+k+1]},$$

which is easily seen to be larger than p. Thus,

$$a\big((n+1)^2+k+1\big) = a\big((n+1)^2+k\big) = 2^{-1} + 2^{-4} + \cdots + 2^{-(n+1)^2}$$

and

$$b\big((n+1)^2+k+1\big) = p\big((n+1)^2+k\big) = 2^{-1} + 2^{-4} + \cdots + 2^{-[(n+1)^2+k+1]},$$

so that $[a(i), b(i)]$ satisfies (*) for $i = (n+1)^2 + k + 1$ in $S(n+1)$. In particular, (*) holds for each $i \in S(n+1)$, and this concludes our induction proof of (*).

Now, using (*) to calculate $p(i)$ for $i = n^2 \in S(n)$ and $p(i)$ for $i = n^2 - 1 \in S(n-1)$, we find

$$\frac{|p(n^2) - p|}{|p(n^2-1) - p|} = \frac{2^{2n}(1 - 2^{-2n} - 2^{-4n-3} - 2^{-6n-8} - \cdots)}{1 + 2^{-2n-3} + 2^{-4n-8} + 2^{-6n-15} + \cdots}.$$

Since this ratio is unbounded for increasing n, the sequence $p(i)$ cannot be linearly convergent!

Nested Polynomials and Efficient Exponentiation Algorithms for Calculators

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Lyle Cook and James McWilliams [TYCMJ 14 (January 1983) 52–54] presented a simple algorithm for approximating the cube root of a number on a calculator equipped with a square root key but with no general power or root key. This stimulated further thinking on how to combine some elementary notions—binary representations and the nested form for polynomials—to produce efficient exponentiation algorithms for calculators with square and square root keys.