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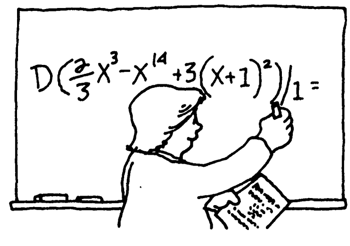

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

## A Diagonal Perspective on Matrices

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How do you think about a matrix? As an abstract linear operator over a field, or over a ring? As a set of row vectors, or column vectors? As a rectangular array of numbers? Most likely you use all of these images and more, depending on the context. Mathematicians are comfortable switching between mindsets as circumstance or convenience demands. This is a skill well worth cultivating in our students, too.

Our purpose here is to describe one such alternative mindset. In an elementary linear algebra course our students learn about *column* and *row* vectors, *column* and *row* spaces, and *row* reduction. This useful and important point of view is encouraged by the traditional, rectangular notation, $A = (a_{ij})_{i,j=1}^{n}$. In this note, however, we will present arbitrary real matrices from a *diagonal* perspective, offering a few simple contexts in which this too is a useful mode of thought.

Let us take $\delta_0$ to be the set of $n \times n$ diagonal matrices, $\delta_1$ the set of superdiagonal matrices, $\delta_{-1}$ the set of subdiagonal matrices, and in general, for $-(n-1) \leq i \leq n-1$, let $\delta_i$ be the set of $n \times n$ matrices whose only nonzero entries are in the $(k, k+i)$ positions. Multiplication of a column vector by an element of $\delta_i$ shifts its entries $i$ places up or down and scales the entries of the shifted vector by the nonzero elements of the matrix. Thus the members of $\delta_i$ are called *shift matrices.*

**Main fact.** *If $A \in \delta_i$ and $B \in \delta_j$, then $AB$ is an element of $\delta_{i+j}$.*

*Proof.* Suppose $AB = (c_{kl})_{k,l=1}^{n}$, where $c_{kl} = \sum_{m=1}^{n} a_{km} b_{ml}$. If $c_{kl}$ is nonzero, then $a_{km}$ and $b_{ml}$ are both also nonzero for some value of $m$. In fact, since $A \in \delta_i$ and $B \in \delta_j$, we have $m = k+i$ and $l = m+j$. Thus $l = k+i+j$ and it follows that $AB \in \delta_{i+j}$. $\square$

Less formally, since the action of an element of $\delta_i$ on a column vector is to shift it, and an arbitrary matrix may be considered a set of column vectors, the proof is complete.

Clearly, if $A$ and $B$ are $n \times n$ matrices, then $A = \sum_{i=1-n}^{n-1} \alpha_i$ and $B = \sum_{i=1-n}^{n-1} \beta_i$, where $\alpha_i, \beta_i \in \delta_i$. As matrix multiplication is distributive, applying our main fact gives us

$$AB = \sum_{i,j=1-n}^{n-1} \alpha_i \beta_j = \sum_{k=1-n}^{n-1} \gamma_k, \quad \text{where} \quad \gamma_k = \sum_{i+j=k} \alpha_i \beta_j \in \delta_k.$$

To see the benefits of thinking of a matrix as a sum of diagonal shift matrices, consider the usual computational proof that the product of two upper triangular matrices is upper triangular. The result is obscured by the difficulty of picturing all possible row-by-column products. However, if $A$ and $B$ are upper triangular, then $A$ and $B$ are sums of upward shifts only; that is, $A = \sum_{i=0}^{n-1} \alpha_i$ and $B = \sum_{j=0}^{n-1} \beta_j$. From above, the shifts $\gamma_k$ that occur in the product $AB$ must also be upward, since $k = i + j \geq 0$.

Here are two exercises, taken from Gene Golub and Charles Van Loan, *Matrix Computations* (Johns Hopkins University Press, Baltimore, 1983). Can you write the solutions using both the traditional notation and our diagonal notation?

**Problem 1.** Show that a strictly upper triangular matrix is nilpotent.

**Problem 2.** Recall that an $n \times n$ matrix is said to have *upper Hessenberg form* if all of the entries below its subdiagonal are zero. Show that if $A \in \mathbb{R}^{n \times n}$ is upper triangular and $B \in \mathbb{R}^{n \times n}$ is upper Hessenberg, then $C = AB$ is upper Hessenberg.

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## Finding a Determinant and Inverse Matrix by Bordering

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Consider the class of $n \times n$ matrices $B$ of the form

$$\begin{bmatrix} b_1 & a_1 & a_1 & \cdots & a_1 \\ a_2 & b_2 & a_2 & \cdots & a_2 \\ a_3 & a_3 & b_3 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_n & a_n & a_n & \cdots & b_n \end{bmatrix},$$

where for all $i$, $a_i \neq b_i$. We wish to find convenient formulas for the determinant and, when appropriate, the inverse of such a matrix. There are, of course, all-purpose methods for solving this problem, but here is an approach that takes advantage of the special nature of these matrices and leads to an elegant result.

**Lemma.** *If $B$ is any invertible matrix and $A = \begin{bmatrix} I & J \\ 0 & B \end{bmatrix}$, where $I$ represents an identity matrix and $J$ is any matrix of appropriate size, then $A$ is invertible and* $A^{-1} = \begin{bmatrix} I & -JB^{-1} \\ 0 & B^{-1} \end{bmatrix}$.