Substituting (1) into (2), we arrive at

$$A = \frac{nr^2 h}{h - 2r} \tan\left(\frac{\pi}{n}\right).$$

The volume of any cone is given by $\frac{1}{3}Ah$, where h is its height and A is the area of its base. Thus, the volume V of our pyramid is given by

$$V = \frac{1}{3}Ah = \frac{nr^2 h^2}{3(h - 2r)} \tan\left(\frac{\pi}{n}\right).$$

Note that the domain of the function $V(h)$ is the open interval $2r < h < \infty$, and the volume increases without bound as h approaches either endpoint of this interval. To minimize V we set the derivative

$$\frac{dV}{dh} = \frac{nr^2}{3} \tan\left(\frac{\pi}{n}\right) \frac{h(h - 4r)}{(h - 2r)^2}$$

equal to zero and find that the minimum volume is attained when $h = 4r$. Somewhat surprisingly, our answer is independent of the number n of sides in the base! Since the minimizing condition $h = 4r$ holds for any value of n, then by letting n go to infinity it follows that the circular cone of minimum volume circumscribed about a sphere of radius r will also have a height equal to $4r$.

For an additional exercise, find the dimensions of the pyramid of minimum surface area whose base is a regular n-gon and whose base and triangular faces are all tangent to a fixed sphere.

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Taylor Polynomials for Rational Functions

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How would you calculate the third-order Taylor polynomial for

$$f(x) = \frac{x^4 + x^2 + 2}{x^3 + x + 1}$$

at the origin? In the usual treatments of Taylor polynomials and Taylor's theorem, rational functions $f(x) = P(x)/Q(x)$ for polynomials P and Q are largely ignored. I imagine this is due to the difficulty of calculating higher-order derivatives of these functions. Here is a way to calculate Taylor polynomials for rational functions that is simple computationally and conceptually. Moreover, as a byproduct we will explicitly calculate the remainders and show that they have a useful form. We can use these explicitly calculated polynomials and associated remainders as concrete examples and motivation for Taylor's theorem.

To find the third-order Taylor polynomial for the above example at the origin, we must first check that the denominator does not vanish at the origin. It does not, so we can proceed with simple long division. For long division of polynomials, we usually arrange the terms of both polynomials in order of decreasing degree, as in the following computation:

$$
\begin{array}{r|l}
 & x \\
\cline{2-2}
x^3+x+1 & x^4+x^2 \qquad +2 \\
 & x^4+x^2+x \\
\cline{2-2}
 & \qquad\quad -x+2
\end{array}
$$

which gives the result

$$f(x) = \frac{x^4+x^2+2}{x^3+x+1} = x + \frac{-x+2}{x^3+x+1}.$$

Our algorithm is similar, but instead we shall write the polynomials so that the degree increases with each additional term, just as if we were dividing power series. This process yields this result:

$$
\begin{array}{r|l}
 & 2-2x+3x^2-5x^3 \\
\cline{2-2}
1+x+x^3 & 2 \quad + x^2 \quad + x^4 \\
 & 2+2x \quad +2x^3 \\
\cline{2-2}
 & -2x + x^2 - 2x^3 + x^4 \\
 & -2x - 2x^2 \quad - 2x^4 \\
\cline{2-2}
 & 3x^2 - 2x^3 + 3x^4 \\
 & 3x^2 + 3x^3 \quad + 3x^5 \\
\cline{2-2}
 & -5x^3 + 3x^4 - 3x^5 \\
 & -5x^3 - 5x^4 \quad - 5x^6 \\
\cline{2-2}
 & 8x^4 - 3x^5 + 5x^6
\end{array}
$$

Thus

$$f(x) = \frac{x^4+x^2+2}{x^3+x+1} = 2-2x+3x^2-5x^3+\left(\frac{5x^2-3x+8}{x^3+x+1}\right)x^4,$$

giving us the expansion

$$f(x) = T_3(x) + R_3(x) \tag{1}$$

for a degree 3 polynomial $T_3(x) = 2-2x+3x^2-5x^3$ and remainder $R_3(x) = (5x^2-3x+8)x^4/(x^3+x+1)$. We shall show that $T_3(x)$ is indeed the third-order Taylor polynomial for f.

There is nothing special about our choice of f or the choice of a third-degree polynomial. Given any rational function $f(x) = P(x)/Q(x)$, we will first check that the denominator does not vanish at the origin, and then begin long division in the manner of power series. Clearly we can continue this process indefinitely, obtaining for each n a degree n polynomial T_n and remainder R_n so that

$$f(x) = T_n(x) + R_n(x), \tag{2}$$

where

$$R_n(x) = \frac{r_n(x)}{Q(x)}x^{n+1}. \tag{3}$$

for some polynomial $r_n(x)$.

To prove that T_n is the Taylor polynomial for f, it is enough to show that $f(0) = T_n(0)$ and that $f^{(k)}(0) = T_n^{(k)}(0)$ for each $1 \le k \le n$. However, the form (3) of R_n

indicates that $R_n(0) = 0$ and $R_n^{(k)}(0) = 0$ for each $1 \leq k \leq n$, thus (2) implies the claim.

So far we have sought to obtain the Taylor polynomials for a rational function f at the origin. For a point a other than the origin, we simply make the change of variables $z = x - a$ and find the Taylor polynomials of this new function at the origin.

These ideas make a good introduction to Taylor polynomials for general functions. I begin by asking my students to use the division algorithm to calculate the Taylor polynomials for a simple function like $f(x) = 1/(1-x)$. Next the students calculate Taylor polynomials about various base points and of various degrees for other rational functions. Students can then graph the original functions and corresponding Taylor polynomials to discover that these polynomials are good approximations of the functions. They might also graph the remainders R_n and their derivatives to discover that all of these functions vanish at the base point of the Taylor polynomial. The form of the remainder, (3), lets students rigorously prove their conjecture. Finally, using the idea that the value of the function and its first n derivatives must agree with the polynomial and its first n derivatives at the base point, students can define and calculate Taylor polynomials for more general functions.

Inquisitive students will want to know if the remainders for Taylor polynomials of general functions behave in the same fashion as the remainders for Taylor polynomials of rational functions; this leads directly to a discussion of Taylor's theorem.

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Pursuit and Regular *N*-gons

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Many readers will recognize the following situation as a generalization of a problem that has appeared in many mathematical textbooks and collections of mathematical puzzles.

> One person is positioned at each vertex of a regular n-gon. Simultaneously, and at the same speed, each person walks directly toward the person who is located k vertices away in the counterclockwise direction. Find an equation that describes the shape of the path taken by each person.

(The case where $n = 4$ and $k = 1$ is the familiar situation.) To approach the generalized problem, begin by positioning a regular "unit" n-gon (i.e., the distance from the center to each vertex is 1) so that the center lies at the origin and a vertex V_0 lies at $(1, 0)$. Therefore, the polar coordinates of the vertex located k vertices counterclockwise from V_0 are $V_k(1, 2k\pi/n)$. If you were to chase someone located *more* than halfway around the n-gon, it would be wiser to move in the direction of $-\theta$; so we impose the restriction that $1 \leq k < n/2$. If n is even, the case where $k = n/2$ makes sense and will be examined separately.

Let P_0 and P_k represent the people positioned initially at V_0 and V_k respectively. Due to symmetry, everyone will travel through the same angle during a given time interval, and at any point they will be the same distance from the center. Therefore, when the polar coordinates of P_0 are (r, θ), those of P_k are $(r, \theta + 2k\pi/n)$. The corresponding rectangular coordinates are $P_0(r\cos\theta, r\sin\theta)$ and $P_k\big(r\cos(\theta + 2k\pi/n), r\sin(\theta + 2k\pi/n)\big)$.