Another Proof of the Irrationality of $\sqrt{2}$

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To this end we shall use the following property of the prime 3:

$$\text{If } a, b \in \mathbb{Z}, \text{ then } 3 \mid (a^2 + b^2) \quad \text{if and only if } 3 \mid a \text{ and } 3 \mid b.$$

This property is clear, since the quadratic residues modulo 3 (that is, the squares mod 3) are 0 and 1. Then, assume, as usual, that $\sqrt{2} = p/q$, where p and q are relatively prime natural numbers. We have $p^2 = 2q^2$ and hence $p^2 + q^2 = 3q^2$. But then $3 \mid (p^2 + q^2)$, and therefore $3 \mid p$ and $3 \mid q$, a contradiction.

Two remarks are in order:

1. The same argument proves the irrationality of $\sqrt{n}$, for any $n \in \mathbb{N}$, $n \equiv 2 (\text{mod } 3)$.

2. The prime numbers p satisfying:

$$a, b \in \mathbb{Z}, \qquad p \mid (a^2 + b^2) \Rightarrow p \mid a \quad \text{and} \quad p \mid b,$$

are exactly those of the form $4m + 3$.

For a similar proof see [Robert Gauntt, The irrationality of $\sqrt{2}$, *American Mathematical Monthly* 63 (1956) 247] or [V. C. Harris, On proofs of the irrationality of $\sqrt{2}$, *Mathematics Teacher* 64 (1971) 19].

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