CLASSROOM CAPSULES

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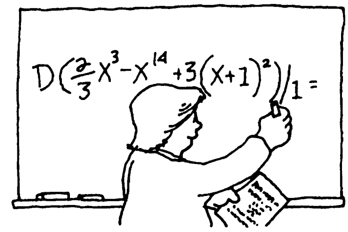

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Tom Farmer.

Computers and Advanced Mathematics in the Calculus Classroom

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As calculus instructors, we often work with classes consisting largely of non-mathematics majors. As a result, we spend a great deal of time and effort constructing examples and projects that demonstrate the uses of calculus in fields other than mathematics. This is great, and as an ex-physicist I'm all for it. However, now I'm a mathematician, so I'd like to propose an alternative: why not develop projects based on advanced mathematics of the sort typically seen in graduate courses?

I can think of at least one good answer to that question: "I have enough trouble getting my students to use the chain rule, and now you want them to prove the Riemann hypothesis?" Well no, not really. However, my experience at the end of the first year of a typical calculus sequence is proof that the idea can work, that students can enjoy it and be motivated by it to study mathematics, and that they can gain significant understanding of topics in calculus in the process. The trick to making it work is to replace the years of "theorem-proof" experience usually prerequisite to studying advanced mathematics with a computer-aided, investigative approach.

As an example, let me briefly describe the final project I assigned my second-semester calculus students in the spring of 1997. As is typical in most calculus sequences, we had covered (among other topics) Riemann integration, the average value of a function, and the limiting behavior of infinite sequences and series. As is also typical, the level of understanding of these topics varied. Looking for a way to inspire some thoughtful consideration to enhance understanding (as opposed to computational facility), I created a graphically-oriented, interactive Mathematica notebook which allowed students to investigate the properties of ergodic averages of functions along sequences in the unit interval $[0,1]$. (I would be glad to e-mail a copy of this notebook to anyone requesting it.)

Such averages take the form $A_n(f,S)=\frac{1}{n}\sum_{k=1}^{n} f(x_k)$, where $S=(x_k)_{k=1}^{\infty}$ is an infinite sequence of numbers from $[0,1]$ and f is a real-valued function. Typically, ergodic theorists ask "Does $\lim_{n\to\infty} A_n(f,S)$ exist?" and "If so, what is it?" For many choices of f and S, the answers are "Yes" and "$\int_0^1 f(x)dx$". This is the content of Birkhoff's Ergodic Theorem, which forms the foundation of the field of ergodic theory.

The historical roots of ergodic theory go back to the great physicist Ludwig Boltzmann (1844–1906), whose Ergodic Hypothesis concerned equilibrium states of physical systems. Let's consider a physical system in equilibrium with its surroundings that is evolving with time. To be specific, let it be a closed container full of gas in thermal equilibrium. Suppose that we insert a small pressure-measuring probe through the wall of the container. Given the wide variety of states available to the system, at one instant a large number of gas molecules could be striking the probe, resulting in a large pressure, while a very short time later the number could be small, resulting in a small pressure. So, we might expect to measure large, rapid fluctuations in pressure, and to get entirely different results if we repeated the experiment tomorrow. However, for gases in equilibrium we know that pressure measurements are quite steady and repeatable. Boltzmann put forth two ideas to explain this. First, he made the reasonable assumption that the time needed to take the measurement is long relative to the fluctuations in the state of the gas, so that what we actually measure is a time average of pressures. This explains why pressure is steady. To explain the repeatability of measurements, he postulated Boltzmann's ergodic hypothesis, which says (very roughly) that over a long period of time, the system will pass through every possible state, and will pass through each state equally often.

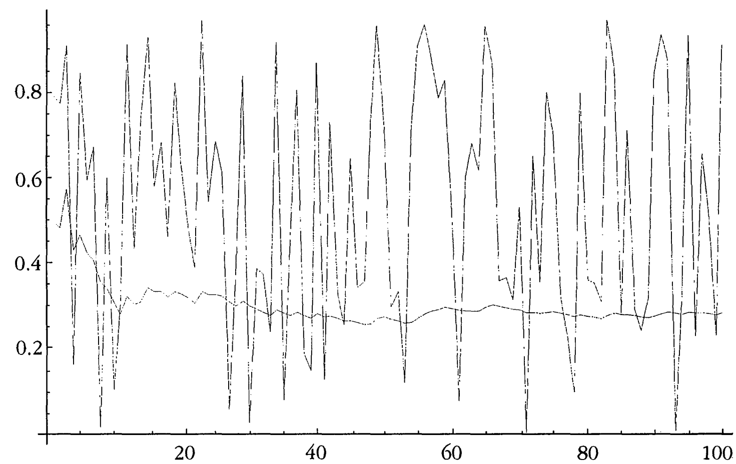
To see how this hypothesis explains the repeatability of measurements, and what it has to do with Birkhoff's Ergodic Theorem, let's consider a much simpler system. Suppose the states of our system can be labeled by the real numbers in $[0, 1]$. As time passes, the evolution of the state of our system corresponds to a time variation of the label attached to the state. We are interested in measuring some property of our system that takes on a definite value in each different state of the system. Let $f(x)$ be the measurement on state x. We make a sequence of measurements at discrete times; at time k, we label the state of the system x_k. Then the long-time average of this sequence of measurements is $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_k)$. Now bring Boltzmann's hypothesis into the picture. As $n \to \infty$, almost all sequences $(x_k)_{k=1}^{n}$ come closer and closer to uniformly filling the entire unit interval. The long-time average then looks like a limit of Riemann sums, with limiting value $\int_0^1 f(x)\,dx$. Since our interval is one unit long, this is the average value of f, which can be interpreted as the average value of the measurement over all possible states of the system. Thus, any measurement performed on a sufficiently long time scale relative to the fluctuations in our system will essentially be an average over all possible states of the system. This not only explains the repeatability of measurements on equilibrium systems, but also leads to the statement of Birkhoff's Ergodic Theorem as we saw it earlier, $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_k) = \int_0^1 f(x)\,dx$. Thus, we see that Birkhoff's Ergodic Theorem is essentially a restatement of Boltzmann's ergodic hypothesis, which became a theorem when Birkhoff proved it.

The proper study of the questions of ergodic theory typically involves a healthy dose of measure theory and functional analysis. Thinking about their answers can lead calculus students to insights into all of the calculus topics mentioned above, as well as such areas as chaotic dynamical systems and Monte Carlo integration.

The Mathematica notebook the students used allowed them to specify a function f and sequence $S = (x_n)_{n=1}^{\infty}$, and then watch a graph of the sequences $S = (x_n)_{n=1}^{N}$ and $(A_n(f, S))_{n=1}^{N}$ unfold for increasing N. This provided a reasonable indication of the existence or non-existence of the limits of the sequences. The students worked in groups of two or three, using the notebook to investigate the answers to several questions:

- How do the properties of f and S relate to the existence of $\lim_{n\to\infty} A_n(f, S)$?
- What can you say about the value of the limit? (Here, experimentation with familiar functions led most students to independently discover Birkhoff's Ergodic Theorem, and gain insights into Riemann integration and the average value of a function.)
- What happens if S is a uniformly distributed sequence of random numbers on $[0,1]$? (Investigation here leads to discovery of the principle of Monte Carlo integration.)
- Suppose S is a sequence of values generated by iteration of the logistic map; i.e., $x_{k+1} = \lambda x_k(1 - x_k)$, with $x_0 \in [0,1]$ specified, and $\lambda \in [0,4]$ an adjustable parameter. How does $\lim_{n\to\infty} A_n(f, S)$ change with λ? As λ increases, S becomes chaotic, so students can gain insight into the consequences of chaotic behavior, and the robustness of system behavior when parameters vary.

Typically, part of a student's investigation procedure might consist of graphically studying $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(x_k)$ in the cases $f(x) = x^n$, $n = 0, 1, 2, 3, \ldots$. In these cases, the limits are discovered to be $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ (See the illustration below for the



First output screen when $f(x) = x^3$.

case $n = 3$). Most students will recognize these values as being $\int_0^1 x^n\, dx$. Further experimentation with other functions leads to the independent discovery of the statement of Birkhoff's Theorem.

Assigning this project was an experiment and, as it turned out, a very successful one. It led to excellent, thoughtful consultations with students and among group members as they worked. With a minimum of hints and nudging on my part, each group produced a written report of their results. The sophistication of their conclusions surprised me, and the enthusiasm with which they discussed them was invigorating. I think the project may have even caused a couple of potential engineers to see the light and consider mathematics as a career, though only time will tell.

Clearly, this experimental approach to investigating advanced concepts and gaining insights into basic concepts has much to recommend it. I can see the same idea being applied in several other freshman and sophomore level courses. For example, in a linear algebra course it could lead students into an experimental investigation of the concepts of functional analysis, or in a multivariable calculus course it could lead to a computer-aided differential geometry project. Such projects could be implemented using any of the existing popular mathematical software packages, or even developed from scratch with computer language compilers. Even the development of such a project could become a project for an advanced student with sufficient computer expertise. I hope others will try these ideas; and if they do, I trust they will experience as much success as I did.

A Natural Proof of the Chain Rule

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The first three editions of Hardy's *A Course in Pure Mathematics* **[4]** contain a "natural" proof of the familiar Chain Rule for differentiating the composition of two real-valued functions of a real variable. Unfortunately, the proof was wrong!

Chain Rule. Let I, J be open intervals of real numbers, $f: I \to J, g: J \to \Re$, f differentiable at c, g differentiable at $f(c)$. Then $g \circ f$ is differentiable at c, and $(g \circ f)'(c) = g'(f(c)) \cdot f'(c)$.

The undergraduate real analysis (or advanced calculus) course allows students to experience the striking power of creating and proving significant results by making natural choices and educated guesses, and then proving these results using a few basic techniques. For the Chain Rule proof, one begins with the definition of a derivative, a familiar technique is applied to transform the problem so the hypotheses can be used, and then the proof follows easily. But then a subtle flaw is revealed, the attempt abandoned, and a special technique is introduced; e.g., **[2–7]**. Often, the motivational step is skipped and the unmotivated proof is directly presented; e.g., **[1]**, **[8]**. In this note, we show how to prove the Chain Rule by following the original path, using techniques familiar to students from previous work.

Natural "proof" of the Chain Rule. Using the definition of the derivative of a composite function,

$$(g \circ f)'(c) := \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c} \tag{1}$$