$b = 0$ (or $a = 0$): if $f_{xx} < 0$ (or equivalently $f_{yy} < 0$) at (x_0, y_0), then we have a maximum, if either (and hence both) is positive, then we have a minimum.

The quantity $D = f_{xy}^2 - f_{xx} f_{yy}$ tells even more than that. If $D > 0$ then the discriminant is positive, meaning that $g''(0)$ can be either positive or negative, depending on a and b. This says that in some directions there is a minimum, and in others a maximum, so that we have a saddle point.

If $D = 0$, then either $g''(0) \equiv 0$, in which case the second derivative test is inconclusive in all directions (a, b) or else $g''(0)$ is semidefinite, e.g. $g''(0) \geq 0$ and there is exactly one direction in which the second derivative test is inconclusive. Thus, if $D = 0$, then we cannot tell without further analysis if (x_0, y_0) is a saddle point or an extremum of some type (though if $g''(0) \geq 0$ as mentioned above, then (x_0, y_0) cannot be a relative maximum).

Notice that we need only the chain rule for multivariate functions to obtain this formula, so its proof need not be relegated to 'advanced calculus' courses. Compare and contrast this with the somewhat less elementary treatments in the references.

One can also use this as a motivation for the study of quadratic forms, diagonalization, eigenvalues, positive definiteness, etc.

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Physical Demonstrations in the Calculus Classroom

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The tremendous success of mathematical modeling is an article of faith among scientists. Indeed, in his well-known paper "The Unreasonable Effectiveness of Mathematics in the Natural Sciences," Eugene P. Wigner expresses a feeling akin to awe at the ability of modern mathematical science to predict as well as to describe empirical events. As calculus instructors wishing to convey a sense of this remarkable interplay between our subject and the "real world," we began looking for ways to elicit more student interest and involvement in our treatment of applications.

The idea we eventually decided to pursue occurred the day when one of us used a glass of water with a pencil standing in it to illustrate Snell's law after deriving it from Fermat's principle. The novelty of this simple physical example linked to calculus drew a favorable response in class and suggested that more ambitious demonstrations might be even more satisfying. The main purpose was to make discussions more memorable rather than to pursue an application area in greater depth; consequently we looked for activities that would involve minimal equipment and class time while inviting hands-on student participation.

The most effective demonstration we have used in calculus classes involves what is known as Torricelli's law, which concerns the rate at which a fluid drains out of a hole in a container. In our experiment, the container is a cylinder with vertical axis and in this case Torricelli's law states that the time rate of change of volume V of water in the draining container is proportional to the square root of the water's depth h. Since the volume of a cylinder is proportional to height, the law reduces to $h'(t) = k\sqrt{h}$.

We have used this demonstration at the point in the Calculus I course when antiderivatives have been introduced and one can solve separable differential equations. The class period begins with a discussion of the question: if water is draining out of a vertical cylindrical tank, will the volume of water in the tank decrease at a constant rate or will the rate vary with time? The class surely will agree that we expect the rate to decrease with time because experience tells us that the stream of water coming out of the hole will be greatest at first when the depth is great and will be reduced to a dribble when the depth is close to zero. Thus, we expect that the linear model $h'(t) = k$ will not very accurately describe this experiment. We intend to compare the linear model with the model given by Torricelli's law.

The following information is then presented to the students:

Problem. A small hole is drilled in the side of a cylindrical container and the height of the water level (above the hole) goes from 10 cm down to 3 cm in 68 seconds. Estimate the height at intermediate times.

The linear model:

$$\frac{dh}{dt} = k,\ h(0) = 10,\ h(68) = 3$$

can be seen to have approximate solution $h(t) = -0.103t + 10$.

The Torricelli model:

$$\frac{dh}{dt} = k\sqrt{h},\ h(0) = 10,\ h(68) = 3$$

has approximate solution $h(t) = 0.00044t^2 + (-0.133)t + 10$.

The table below gives the values of h predicted by each of these models for various intermediate times. We will fill in the column of observed values when we perform the experiment.

time t	linear h	Torricelli h	observed h
0	10.0	10.0	
10	9.0	8.7	
20	7.9	7.5	
30	6.9	6.4	
40	5.9	5.4	
50	4.8	4.4	
60	3.8	3.6	
68	3.0	3.0	

The equipment for this demonstration is easily obtained at little or no cost. We prepare a two-liter clear plastic soft drink bottle, whose midsection is essentially cylindrical, by drilling a clean 4 millimeter hole near the bottom of the cylindrical part. We also attach to the bottle a strip of masking tape with centimeters marked on it and zero corresponding to the top of the hole. We bring this bottle to class along with another bottle of water (as a water source), a laundry bucket (as a water sink) and a board to span the bucket. We can probably always rely on a student to have a digital watch to keep time but, to be safe, we bring such a watch. In advance of class time, we run through the process several times to establish the time required to drain the cylinder; with our apparatus it consistently took 68 seconds for the water depth above the hole to drop from 10 centimeters to 3 centimeters. Given this, the problem is to predict what the depth will be at times 10, 20, 30, 40, 50, and 60 seconds.

A vital part of the demonstration is to have students participate. We have had no difficulty in getting three volunteers to play all the roles. The timekeeper helps out initially by pouring the water into the leaky bottle while the bottle keeper covers the hole with a finger. Meanwhile, the recorder copies the table of predicted values on the blackboard. As the experiment progresses, the time keeper calls out $\dots 8, 9, \mathbf{10}, \dots, 18, 19, \mathbf{20}, \dots$ and at the appropriate times the bottle keeper estimates out loud the depth reading which can be done with accuracy within a tenth of a centimeter. This person should be cautioned not to look at the predicted values in order to avoid being influenced by them. As the depths are called out the recorder records them on the blackboard. Our experience in trying this in three different classes was that the experimental data were virtually identical to the Torricelli predictions, making for an impressive and memorable demonstration.

A second demonstration we have used involves less apparatus but one item that must be made in advance: a thin "plate" shaped like the interior of a parabola from the vertex back to a line perpendicular to the axis. The basic idea is to determine from first principles, via integral calculus, just where the balance point ought to be. The chance for people to test the result on a physical model gives a "payoff" to the example.

We are still experimenting with ideas along the lines described above. Such topics as Newton's law of cooling, period of a pendulum, differentials for inverse square laws, and spark tapes for velocity/acceleration seem promising, if one can find the right mix of calculus and physical interaction in the classroom.

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Rubberbanding and Holding Out

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In discrete mathematics, one problem of interest is counting bit strings that have some particular property. Two such problems deal with the number of ways that zeros can appear together and the number of ways that they can be separated. Specifically:

Problem 1. How many eight-bit strings with exactly two zeros are there in which the two zeros appear together?

Problem 2. How many eight-bit strings with exactly two zeros are there in which the two zeros *do not* appear together?

To solve the first problem, we "rubberband" the two zeros together and then count the number of ways that six ones can be arranged with the zeros: $C(7,1)$ ways. To solve the second problem, we may use a complement approach. That is, subtract the number of ways the zeros can appear together from the total number