and continuing in this fashion,

$$J_{r-1} = q^{k+1} J_{r+k} + \sum_{i=0}^{k} (r+i) q^i$$

for values of k from 0 up to where $J_{r+k} = 0$. Setting $r = 1$ and assuming k can be chosen so that $J_{k+1} = 0$ (which makes $k+1$ the day that the last candies are eaten), we obtain

$$J_0 = q^{k+1} J_{k+1} + \sum_{i=0}^{k} (1+i) q^i = 0 + \frac{d}{dq} \sum_{i=0}^{k} q^{i+1} = \frac{d}{dq}\left(\frac{q^{k+2} - q}{q-1}\right).$$

It follows that the original number of candies must be given by

$$J_0 = \frac{(k+1)q^{k+2} - (k+2)q^{k+1} + 1}{(q-1)^2}. \tag{1}$$

If $k = 0$ (so that $J_1 = 0$), this gives the trivial solution to the problem: $J_0 = (q^2 - 2q + 1)/(q-1)^2 = 1$. On the other hand, if $k > 0$ (and $J_{k+1} = 0$), then we must consider how equation (1) can yield an integer value for J_0. Note that $q = n/(n-1)$ implies $q - 1 = 1/(n-1)$, so (1) becomes (after algebraic simplification)

$$J_0 = \left(\frac{n}{n-1}\right)^k n(k+2-n) + (n-1)^2. \tag{2}$$

Since n and $n-1$ are relatively prime, the right-hand side of (2) is an integer only when $n = k+2$. Consequently, day $k+1$ when the candy is finished is day $n-1$, and the original number of candies is $(n-1)^2$, as claimed. One can check that this value for J_0 does indeed lead to an integer sequence $\{J_r\}$. In fact, it is easily proved by induction that $J_r = (n-1)(n-r-1)$ for $r = 0, \ldots, n-1$.

As an exercise, here is a related problem that can be solved by similar means:

> On the rth day, a pool of I dollars is increased by r^2 and then reduced by $s\%$ (where $n = 100/s$ is an integer). Assume that the pool has an integer number of dollars and that on some future day the original amount I is restored. Show that $I = (n-1)^2(2n-1)$ is the unique nontrivial solution under these conditions and that, in this case, $2(n-1)$ is the number of days required.

———— o ————

A Simple Solution of the Cubic

Dan Kalman (kalman@email.cas.american.edu), American University, Washington, DC 20016
James White (mathwrig@gte.net), Mathwright Library, Marina, CA 93933

The quadratic formula for the general degree-two equation is one of the most familiar equations in mathematics. Surely every college mathematics teacher can quote it and derive it without effort. In contrast, the corresponding equation for the solution of the general cubic is quite obscure. We are all aware that such a formula exists, but it is an uncommon mathematician who can quote the result, let alone derive it from first principles. Imagine our surprise, therefore, at discovering a simple algebraic derivation in the middle of looking for something else. Even more surprising, when we reviewed the literature, we discovered (or rediscovered) other derivations that are just as simple. Indeed, Oglesby [**9**] came up with a closely related approach 75 years ago. In retrospect, the solution of the cubic seems direct enough that we ought to have been more familiar with it. We hope you will experience a similar reaction as we share the derivation we found so serendipitously and sketch the more usual approach.

Before proceeding, we should recall that an arbitrary cubic equation can be reduced to one of the form

$$x^3 + px + q = 0 \tag{1}$$

by a linear change of variable. So in what follows, we will consider only this kind of cubic equation.

The derivation that we will present depends on the following identity:

$$(\omega a + b + c)(a + \omega b + c)(a + b + \omega c) = (a^3 + b^3 + c^3)\omega - 3abc\omega^2. \tag{2}$$

Here, a, b, and c are arbitrary complex constants and $\omega = \left(-1 + i\sqrt{3}\right)/2$ is a cube root of 1, so it satisfies a number of identities:

$$\begin{aligned} \omega^3 &= 1, \\ \omega^2 + \omega + 1 &= 0, \\ \omega + 1 &= -\omega^2. \end{aligned}$$

To verify (2), simply multiply out the left side, collect like monomials in a, b, and c, and apply the identities for ω listed above. Symmetry simplifies the process considerably. Collecting together all terms involving a^2b results in a coefficient of $\omega^2 + \omega + 1 = 0$. By symmetry, the terms involving $a^2c, b^2c, \ldots,$ also vanish. In the expansion of the left-hand side of the identity, that leaves only terms involving a^3, b^3, c^3, and abc; when we consider the coefficients of these terms, (2) is easily established.

We discovered identity (2) while working on a problem posed in *Math Horizons*: Solve $\sqrt[3]{x+a} + \sqrt[3]{x+b} + \sqrt[3]{x+c} = 0$. Although (2) turned out to be irrelevant for that problem, the identity was so appealing that we were motivated to seek another use for it. In the process, we stumbled on the following simple solution of the general cubic equation.

To render the identity more recognizable, let's replace a with x, which is to be thought of as the variable of the cubic. That produces

$$(\omega x + b + c)(x + \omega b + c)(x + b + \omega c) = (x^3 + b^3 + c^3)\omega - 3xbc\omega^2.$$

Factoring out ω on the right and rerranging the remaining terms leads to

$$(\omega x + b + c)(x + \omega b + c)(x + b + \omega c) = \omega(x^3 - 3xbc\omega + b^3 + c^3). \tag{3}$$

Now we can recognize that the right side is essentially the same as what appears in (1), provided that the following relations hold:

$$-3bc\omega = p, \tag{4}$$

$$b^3 + c^3 = q. \tag{5}$$

Given values of p and q, we need only determine a b and c satisfying these relations, whereupon (3) provides a factorization to linear factors. Fortunately, we can solve for b and c in a straightforward way. Indeed, let's just rewrite the original system of equations in the following form:

$$b^3c^3 = -\frac{p^3}{27},$$
$$b^3 + c^3 = q.$$

It is immediately apparent that b^3 and c^3 are the roots of the quadratic equation $x^2 - qx - (p^3/27) = 0$, and they are given by

$$\left(\frac{q \pm \sqrt{q^2 + (4p^3/27)}}{2}\right)^{1/3}.$$

At this point it is tempting to write

$$b = \left(\frac{q + \sqrt{q^2 + (4p^3/27)}}{2}\right)^{1/3},$$
$$c = \left(\frac{q - \sqrt{q^2 + (4p^3/27)}}{2}\right)^{1/3}. \tag{6}$$

However, that will not necessarily produce solutions consistent with equation (4). In particular, when p and q are real, we obtain real values for b^3 and c^3 just when $q^2 + (4p^3/27) \geq 0$. In this case, simply extracting cube roots produces real values of b and c, which evidently can satisfy (4) only in the exceptional case $p = bc = 0$. Thus, a bit more care is required.

Using (6) as expressions for b^3 and c^3, we have three complex cube roots among which to choose b and c. Clearly, (5) will be satisfied no matter how we choose these cube roots. To satisfy (4) as well, we simply choose any of the three complex cube roots for b and then define c as $-p/(3\omega b)$.

To complete the solution of the cubic, we note that the solutions to (1) must also be roots of

$$(\omega x + b + c)(x + \omega b + c)(x + b + \omega c) = 0.$$

By inspection, the solutions are

$$x = -\frac{b+c}{\omega}, \qquad x = -(\omega b + c), \qquad x = -(b + \omega c).$$

This result is closely related to, but slightly different from the standard solution to the cubic that has been handed down with little if any modification since it was published by Cardano in 1545. Although it was originally derived by a different method, Cardano's solution can be formulated in terms of the following identity:

$$(a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c) = a^3 + b^3 + c^3 - 3abc. \tag{7}$$

This identity has appeared in earlier papers **[6, 9]** on the solution of cubic equations. It is very similar to (2) and can be derived from (2) by replacing a with a/ω. From (7), virtually the same steps presented above lead to the traditional form of Cardano's solution to the cubic. The symmetry of (2) may make its verification somewhat simpler than the verification of (7). Otherwise, either identity provides a simple approach to solving the cubic.

Here is a streamlined version of the Cardano solution, which is particularly simple and memorable. Following the presentation in [**10**], begin with the cubic in the form $x^3 = px + q$ and replace x with $b + c$. That leads directly to the equation $3bcx + b^3 + c^3 = px + q$. Now match the coefficients of x; that is, make $3bc = p$ and $b^3 + c^3 = q$. These conditions are virtually identical to those used earlier, and they allow us to find b and c in terms of p and q. From this point on, the derivation is essentially the same as what was presented before.

The literature on solving the cubic is large; for representative samples, see [**3, 4, 9, 11**]. Besides the presentation in [**10**] previously cited, a recently published version is in [**5**]. A translation of Cardano's solution appears in [**2**], and the history of Cardano's publication and his dispute with Tartaglia [**1, 8**] is quite interesting. Kleiner [**7**] provides a nice discussion of the role of the solution of the cubic in the development of complex numbers.

References

1. Carl Boyer, *A History of Mathematics*, 2nd ed. (rev., Uta C. Merzbach), Wiley, New York, 1991, pp. 282–286.
2. Ronald Calinger, ed., *Classics of Mathematics*, Moore Publishing, Oak Park, IL, 1982, pp. 235–237.
3. H. B. Curtis, A derivation of Cardan's formula, *American Mathematical Monthly* 51:1 (1944) 35.
4. Orrin Frink, Jr., A method for solving the cubic, *American Mathematical Monthly* 32:3 (1925) 134.
5. Jan Gullberg, *Mathematics from the Birth of Numbers,* Norton, New York, 1997, p. 318.
6. Morton J. Hellman, A unifying technique for the solution of the quadratic, cubic, and quartic, *American Mathematical Monthly* 65:4 (1959) 274–276.
7. Israel Kleiner, Thinking the unthinkable: The story of the complex numbers (with a moral), *Mathematics Teacher* 81 (1988) 583–592. Reprinted in Frank J. Swetz, ed., *From Five Fingers to Infinity*, Open Court, Chicago and La Salle, IL, 1994, pp. 711–720.
8. Victor J. Katz, *A History of Mathematics: An Introduction*, Harper Collins, New York, 1993, pp. 330–334.
9. E. J. Oglesby, Note on the algebraic solution of the cubic, *American Mathematical Monthly* 30:6 (1923) 321–323.
10. John Stillwell, *Mathematics and Its History*, Springer-Verlag, New York, 1989, p. 55.
11. Robert Y. Suen, Roots of cubics via determinants, *College Mathematics Journal* 25:2 (1994) 115–117.

MATH and Other Four-Letter Words

Marc D. Sanders (sanders@dickinson.edu) and Barry A. Tesman (tesman@dickinson.edu), Dickinson College, Carlisle, PA 17013

Once parametric equations have been introduced in a calculus course, and after the students have graphed a few examples and seen the usual methods for parametrizing line segments and circular arcs, then the class is ready for this "spelling" lab. Working with the letters of the alphabet is an amusing and engaging way for students to become confident about writing parametric equations and plotting parametric curves.