It is interesting that the terms in the sequence $\{r_n(t)\}$ are all continuous functions whose integrals over $[0,1]$ equal I, and from above their graphs all intersect the horizontal line $y=I$. Moreover, provided that the mesh of the partitions $a=x_0, x_1, \dots, x_n=b$ approaches zero as $n\to\infty$, the sequence $\{r_n(t)\}$ converges *uniformly* on $[0,1]$ to the constant function I, since all Riemann sums of f relative to partitions with sufficiently small mesh differ from I by as little as we please. Figure 1, produced using *Mathematica*, shows the graphs of the constant function $I=\int_1^4(e^x/x)\,dx \approx 17.736$ and the functions $r_{10}(t), r_{20}(t), r_{30}(t)$ on $[0,1]$, using partitions of $[1,4]$ into equal subintervals. The features just mentioned of the sequence $\{r_n(t)\}$ are apparent in the figure. Do you see from the graph that the midpoint sums give a good estimate of this integral?

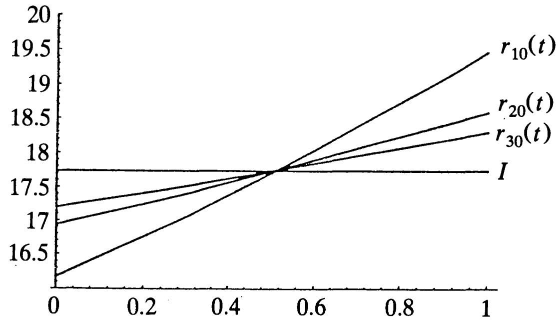

Figure 1

Card Shuffling in Discrete Mathematics

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The problem of determining the number of perfect shuffles required to return a deck of cards to its original order is rich with ideas from elementary algebra, discrete mathematics, and number theory [**1**]. Herstein and Kaplansky [**2**] used card shuffling to motivate a discussion of permutation groups. We have found that for beginning students in discrete mathematics, card shuffling can bring to life the abstract notions of relations and digraphs.

If the overhand shuffle (described below) is applied repeatedly to a deck with an even number of cards, eventually the deck will return to its original order. Let $f(n)$ denote this fundamental period of the shuffling process applied to a deck of $2n$ cards. Although a closed formula for $f(n)$ has yet to be found, we shall see that $f(n)$ is easily computed because it has a simple algebraic interpretation. For more advanced students in a number theory or abstract algebra course, the computation of $f(n)$ is seen to involve the structure of the multiplicative group of units in the ring of integers modulo $2n-1$.

The overhand shuffle. Cut the deck of $2n$ cards (Figure 1a) into two equal stacks, say with the upper half placed on the right as in Figure 1b. Beginning with the stack on the right, the top card is alternately drawn from each pile and placed in a third pile (Figure 1c) until the two stacks are exhausted. Note that this shuffle interchanges the top and bottom cards of the deck, so clearly $f(n)$ is always even.

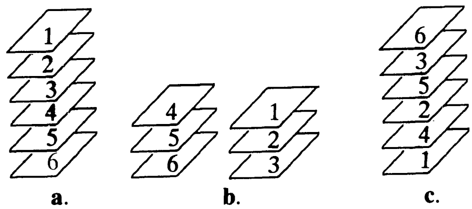


Figure 1

We have found that "hands-on" experience determining $f(n)$ for small values of n gets students in our discrete mathematics class involved with the problem. Working with a partner, the students count shuffles for decks of two to 16 cards and make predictions along the way. At first it seems that a simple pattern will emerge: up to 14 cards, most decks require two shuffles less than the number of cards, the exceptions being two shuffles for two cards and six shuffles for 10 cards (see Table 1). Upon discovering that a 16-card deck requires only four shuffles to return to its original order, the students are surprised and demand an explanation.

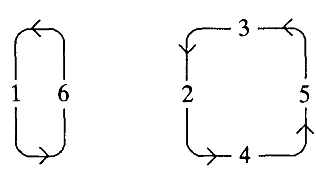
Table 1

Number of cards, $2n$	Number of shuffles, $f(n)$	Cycle lengths in R
2	2	2
4	2	2, 1, 1
6	4	4, 2
8	6	3, 3, 2
10	6	6, 2, 1, 1
12	10	5, 5, 2
14	12	12, 2
16	4	4, 4, 4, 2, 1

The first step is to analyze a single shuffle by writing down the resulting card positions as in Figure 1. It can be seen that card 6 is sent to the first position, card 3 to the second position, and so on. This gives rise to a relation R on the set of card positions as follows: (i, j) is in R if the card in position i is sent to position j by the shuffle. For example, for $2n = 6$,

$$R = \{(6,1),(3,2),(5,3),(2,4),(4,5),(1,6)\}.$$

In the discrete mathematics course, relations on a set are given visual form by directed graphs (digraphs). Representing each card position by a node, an arrow is drawn from the card's initial position to its position after the shuffle:



Since the shuffle relation R is a one-to-one function (permutation) of card positions, the graph's components are cycles—thus we arrive at the usual cycle decomposition of a permutation. By recording the cycle lengths for the shuffle permutation, as in Table 1, the students can look for patterns.

Some students discover that the number of shuffles required to restore the deck is always the same as the length of the longest cycle, or twice this length. Others may discover that each cycle length is a divisor of the longest cycle length, except possibly the 2-cycle determined by the alternation of the top and bottom cards. In any case, with or without making those two discoveries, it is not hard to see that $f(n)$ is the least common multiple (lcm) of the cycle lengths. Like runners on tracks of different lengths, the cards move through their cycles of positions in the deck; the lcm of the cycle lengths gives the first time all cards are back in their starting positions.

Working in groups, the students can construct the shuffle digraphs for decks up to size $2n = 100$ and determine the corresponding value for $f(n)$ as the lcm of the cycle lengths. For larger decks it is more expedient to simulate the shuffle using a computer. Such a program is easy to write once a general formula for the shuffle permutation R is found. By examining the permutations in a few cases a clear pattern emerges (see Table 2).

Table 2

6 cards	8 cards	10 cards	$2n$ cards
$1 \to 6$	$1 \to 8$	$1 \to 10$	$1 \to 2n$
$2 \to 4$	$2 \to 6$	$2 \to 8$	$2 \to 2n-2$
$3 \to 2$	$3 \to 4$	$3 \to 6$	·
$4 \to 5$	$4 \to 2$	$4 \to 4$	·
$5 \to 3$	$5 \to 7$	$5 \to 2$	$n \to 2$
$6 \to 1$	$6 \to 5$	$6 \to 9$	$n+1 \to 2n-1$
	$7 \to 3$	$7 \to 7$	$n+2 \to 2n-3$
	$8 \to 1$	$8 \to 5$	·
		$9 \to 3$	·
		$10 \to 1$	$2n \to 1$

On each half of the set of card positions the permutation is a linear function with slope -2. Students can apply the point-slope formula on each half to obtain a formula for R:

$$R(i) = \begin{cases} 2n - 2(i-1) & 1 \le i \le n \\ 1 - 2(i-2n) & n+1 \le i \le 2n \end{cases} \tag{1}$$

A computer program can now be designed that uses a two-dimensional array and (1) to trace the cycles for any deck size (up to *maxint*).

Students familiar with modular arithmetic can continue farther still. The two formulas in (1) could be combined into one simple formula if only it were true that $2n - 2(i-1) = 1 - 2(i-2n)$, that is, if only somehow $2n = 1$. We can make this true by working modulo $2n-1$. This effectively identifies the top and bottom positions in the deck, but we observed earlier that the cards in these positions are simply exchanged by the shuffle. Thus we view the stack of card positions as a model of Z_{2n-1}, labeling the residue classes $1, 2, 3, \ldots, 2n-1$, rather than the usual $0, 1, 2, 3, \ldots, 2n-2$, and we consider the residue class 1 to correspond to both the top and bottom card positions. Then formula (1) becomes simply

$$R(i) \equiv -2i + 3 (\bmod 2n-1). \tag{2}$$

This, together with $R(1) = 2n$ and $R(2n) = 1$, determines the permutation R on the complete set of $2n$ card positions.

Now, from (2) it is easy to prove by induction that for all $k \geq 1$,

$$R^k(i+1) \equiv (-2)^k i + 1 (\bmod 2n-1).$$

Thus $f(n)$, the smallest integer k such that R^k is the identity permutation, is the smallest *even* integer such that $i + 1 \equiv (-2)^k i + 1 (\bmod 2n-1)$ for all i. It follows that

$$f(n) = \min\{2k | (-2)^{2k} \equiv 1 (\bmod 2n-1)\} = 2 \min\{k | 4^k \equiv 1 (\bmod 2n-1)\},$$

which is twice the order of 4 in the multiplicative group in Z_{2n-1}. Using this characterization, a computer program can be written to compute $f(n)$ directly for decks of any even size (up to *longint*).

References

1. P. Diaconis, R. L. Graham, and W. M. Cantor, The mathematics of perfect shuffles, *Advances in Applied Mathematics* 4 (1983) 175–196.
2. I. N. Herstein and I. Kaplansky, *Matters Mathematical*, Harper & Row, New York, 1974.

Finding Volumes with the Definite Integral: A Group Project

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The following group project for a first-year calculus course uses enjoyable hands-on experience and "real world" equipment to help students understand the calculus methods used in finding the volume of a given solid. Before doing the project, in a previous class we set up the definite integral for the volume of a square-based pyramid and the integral for the volume of a solid generated by revolving a line segment about the x-axis. Then for homework the students do some standard problems—volumes of solids of revolution and a straightforward volume of a solid by slices. Two days later I bring the project equipment into class (see Figure 1):

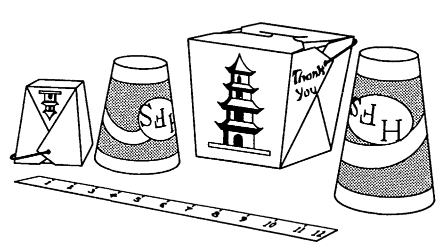

Figure 1