An Example Demonstrating the Fundamental Theorem of Calculus

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When we introduce the Fundamental Theorem of Calculus, it is natural to begin with some simple examples that demonstrate its validity and may be verified without appealing to the theorem itself. The standard examples are usually constant functions and linear functions, for which the geometric area beneath these graphs may be easily determined without computing limits. While in some sense the example of a constant function shows what makes the theorem hold in greater generality, it may seem rather special to beginning students. Here is a function whose graph does not consist of straight lines, for which the theorem may be independently confirmed.

Let $f(x) = \sqrt{1-x^2}$, on the interval $[-1,1]$, whose graph is the upper half of the unit circle. We will show that the conclusion of the Fundamental Theorem of Calculus holds for f, on the interval $(-1,1)$. That is, if $A(t)$ is the area of the region $R(t)$ under the graph from $x=-1$ to $x=t$, then a nice computation will verify that $A'(t) = f(t)$.

To do this, we think of $R(t)$ as the union of a sector of the unit circle and a triangle, as shown in Figure 1. We will accept that the area of a triangle is one-half its base times its altitude, and that the area of an angular sector of a disc of radius 1 with central angle θ (in radians) is $\theta/2$. (It is worthwhile to think about how the calculus definition of these areas relates to other derivations.)

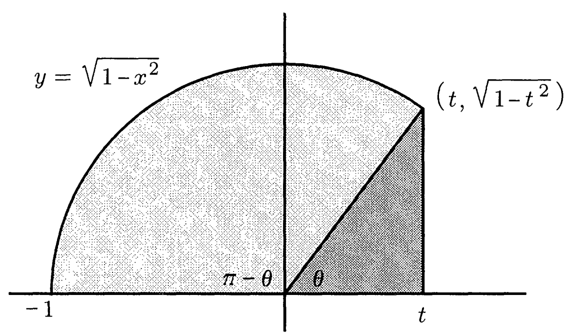

Figure 1. The region $R(t)$.

Since the angle in the triangle is θ, the angle of the shaded sector is $\pi - \theta$, so

$$A(t) = \frac{\pi - \theta}{2} + \frac{1}{2}t\sqrt{1-t^2}$$

where θ satisfies $\cos\theta = t$ (or $\theta = \arccos(t)$). Observe that this equation for $A(t)$ is valid for both positive and negative values of t, since for $t < 0$, $A(t)$ is the difference of the areas of the sector and the triangle, and this is accounted for by the factor of t in the second term on the right side.

Taking the derivative, we get

$$A'(t) = -\frac{1}{2}\frac{d\theta}{dt} + \frac{1}{2}\left(1 \cdot \sqrt{1-t^2} + t \cdot \frac{1}{2}\frac{1}{\sqrt{1-t^2}} \cdot -2t\right).$$

We apply implicit differentiation or the inverse function rule to the defining equation for θ, which gives

$$-\sin\theta\frac{d\theta}{dt} = 1.$$

Then, using $\sin\theta = \sqrt{1-t^2}$ (from the figure or from $\sin\theta = \sqrt{1-\cos^2\theta}$), we arrive at

$$-\frac{d\theta}{dt} = \frac{1}{\sin\theta(t)} = \frac{1}{\sqrt{1-t^2}}.$$

Simplifying and combining, we find

$$A'(t) = \frac{1}{2}\left(\frac{1}{\sqrt{1-t^2}} + \frac{1-t^2}{\sqrt{1-t^2}} - \frac{t^2}{\sqrt{1-t^2}}\right) = \frac{1-t^2}{\sqrt{1-t^2}} = \sqrt{1-t^2} = f(t),$$

just as promised by the Fundamental Theorem of Calculus.

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More Coconuts

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The linear Diophantine problem "Dividing Coconuts," solved by Singh and Bhattacharya [**2**], is thought-provoking and prompted me to find several other solutions. Here is my favorite one. Recall that the problem, first posed by Paul Halmos [**1**], is stated as follows:

> After gathering a pile of coconuts one day, five sailors on a desert island agree to divide them evenly after a night's rest. During the night one sailor gets up, divides the nuts into five equal piles with a remainder of one, which he tosses to a conveniently nearby monkey, and, secreting his pile, mixes up the others and retires. The second sailor does the same thing, and so do the third, the fourth, and the fifth. In the morning the remaining pile of coconuts (less one) is again divisible by 5. What is the smallest number of coconuts that the original pile could have contained?

Why should the monkey be getting coconuts that the sailors worked to gather? Since the original number N has a remainder of 1 upon division by 5, let us suggest that the sailors add four big round stones to the pile and pretend they are coconuts. Now the number is $N+4$ and it is divisible by 5.

When the first sailor gets up in the night and removes one-fifth of the pile, taking no stones, he gets to keep the coconut that would otherwise have been thrown to the monkey. The remaining $\frac{4}{5}(N+4)$ coconuts and stones in the pile is still divisible by 5 because it still contains the four stones. In the same way, the other sailors get up in the night. Each removes a fifth of the existing pile, taking no stones and keeping the coconut that would have gone to the monkey.

By morning, the pile contains $\left(\frac{4}{5}\right)^5(N+4)$ coconuts and stones. Since this number still includes the four stones, it is still divisible by 5, so it must have the form $5m$