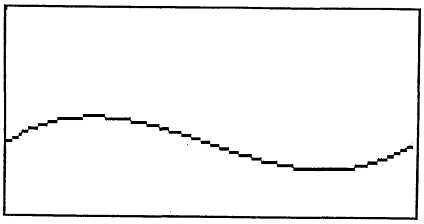

Figure 7

TI-85 graph of $f(x) = (x-2)^3 - x + 3$ on [1, 3].

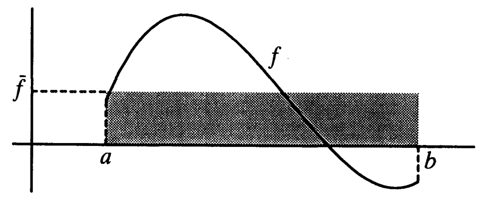

Figure 8

Figure 8? The requirement is that $(\bar{f})(b-a) = \int_a^b f(x)\,dx$, so the answer is YES:

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Now we see that the fundamental theorem of calculus is just a generalization of the point-average slope formula (2), from step functions to arbitrary piecewise continuous functions: *If f is a piecewise continuous function on* $[a,b]$ *and F is a continuous antiderivative of f on* $[a,b]$ *then*

$$F(b) - F(a) = (\bar{f})(b-a) = \int_a^b f(x)\,dx. \qquad (3)$$

Besides the pedagogical benefit of tying the fundamental theorem of calculus to ideas already familiar to students, this approach has two other merits. First, it introduces the fundamental theorem of calculus for piecewise continuous functions, rather than the more limited case of continuous functions. Exercises with step functions and their continuous piecewise linear antiderivatives, such as Exercise 5, can introduce the idea in a simple setting. Second, it is important for students to learn to picture the derivative as an instantaneous rate of change, a *local average velocity*, not just as the local slope of a graph. Students all know that $\Delta x = v\Delta t$ if the velocity is constant; thus the generalization $\Delta x = \bar{v}\Delta t = \int_{t_0}^{t_1} v(t)\,dt$ for a variable velocity function $v(t)$ is very natural, and this is just our formula (3) in a different setting. This approach to the fundamental theorem of calculus provides an interpretation of integration as *transforming a varying local average* $f(x)$ *on* $[a,b]$ *into a global average over this interval*:

$$\bar{f} = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{F(b)-F(a)}{b-a}.$$

This point of view can be helpful in understanding other applications of integrals.

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Chebyshev's Theorem: A Geometric Approach

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Although Chebyshev's theorem is stated in almost all elementary statistics textbooks, few include a proof. The reason is that the usual algebraic proof is not very illuminating to students at this level. Perhaps the geometric approach in this note would help to clarify the proof, at the same time giving beginning students a better intuitive feel for the concepts of variance and standard deviation.

Consider a set of numerical data arranged, purely for convenience, in ascending order, $x_1, x_2, \ldots, x_n$. The average, or *mean*, of this data set is defined to be

$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i$$

and the variance is

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2.$$

Let's take a look at this last formula. Each term $(x_i - \mu)^2$ can be pictured as a square whose side length is $|x_i - \mu|$, the distance between the ith data value and the mean. We will refer to these squares as *tiles*, denoting by T_i the area of the tile associated with the data value x_i. Thus $\sigma^2 = (1/n)\sum_{i=1}^{n} T_i$, which means that the variance may be thought of as the *average-sized tile*. The standard deviation σ of our data set is then *the length of the side of the average-sized tile*. By drawing the tiles associated to a data set, as in Figure 1, a student can visually estimate the average-sized tile and thus can roughly approximate the variance and standard deviation. Note that the combined area of n average-sized tiles equals the total area of all the tiles; that is, $n\sigma^2 = \sum_{i=1}^{n} T_i$. This seemingly innocuous fact will play a key role in our proof of Chebyshev's theorem.

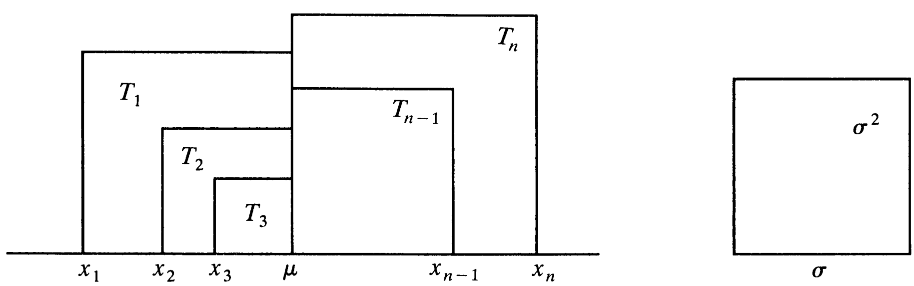

Figure 1
A typical data set, with the associated tiles, and the average-sized tile.

The geometric representations of the variance and standard deviation make it easy to see how these descriptive parameters measure the dispersion of a set of data. If all the data are bunched up near the mean, it is clear that the average-sized tile will be small and, consequently, so will its side length. But if even a small proportion of the data lies far from the mean, the average-sized tile may be rather large. Chebyshev's inequality just makes this qualitative observation a bit more precise.

Chebyshev's theorem. *The proportion (or fraction) of any set of data that lies farther than k standard deviations from the mean is* never more than $1/k^2$, *for any positive integer k.*

Thus if the data set contains n elements, Chebyshev's theorem guarantees that at most n/k^2 of these data values lie farther than k standard deviations away from μ. Suppose for a moment that we have a data set of n elements for which this is not the case, for some particular k. Call the elements that lie farther than k standard deviations away from the mean *outsiders*. Since every outsider lies at a distance greater than $k\sigma$ from the mean μ, each tile associated with an outsider has area greater than $(k\sigma)^2 = k^2\sigma^2$. We have assumed the existence of more than n/k^2 outsiders, so it follows that the combined area of the outsider tiles must exceed $(n/k^2)(k^2\sigma^2) = n\sigma^2$. But this is impossible—the total area of *all* the tiles is $n\sigma^2$. This contradiction proves Chebyshev's theorem.

In short, Chebyshev's theorem says all data sets are xenophobic—they cannot allow too many outsiders, lest the outsiders occupy too much of the total tiled area.

Pizza Combinatorics

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Customer: So what's this new deal?
Pizza Chef: Two pizzas.
Customer: [*Towards four-year-old boy*] Two pizzas. Write that down.
Pizza Chef: And on the two pizzas choose any toppings—up to five [*from the list of* 11 *toppings*].
Older Boy: Do you...
Pizza Chef: ...have to pick the same toppings on each pizza? No!
Four-year-old Math Whiz: Then the possibilities are endless.
Customer: What do you mean? Five plus five are ten.
Math Whiz: Actually, there are 1,048,576 possibilities.
Customer: Ten was just a ballpark figure.
Old Man: You got that right.

On November 8, 1993, this popular commercial, "Math Whiz," first aired on national television. Probably some viewers dimly recalled from their mathematical studies that the large number of possibilities has something to do with permutations, factorials, combinations, or some other long-forgotten technique, but perhaps only the authors of this article were so intrigued that they investigated whether the four-year-old "math whiz" was actually correct in his calculations.