Fanning out. Here are some suggestions and alternatives, their usefulness depending on the abilities of the students involved.

If you leave it to the student to choose f (as in the sample instructions), do not waste many interesting examples such as $r = 1 + \sin\ \theta$ by showing them as finished fans. The ellipse $r = 6/(2 + \cos\ \theta)$ is a good example to show. The circle (or one-leafed rose) $r = \sin\ \theta$ and the four-leaved rose $r = \sin\ 2\theta$ (which produces only two leaves here) nicely show the result of deleting all points with negative radii.

You might consider how to use "overlay" features to handle the possibility that $f(\theta) < 0$, for which an overlay of $r = -f(\theta - \pi)$ is often successful. Or, if more than just the interval $0 \le \theta \le 2\pi$ is involved in a polar sketch, overlays of $r = f(\theta - 2\pi)$, $r = f(\theta + 2\pi)$, $r = f(\theta - 4\pi)$, and so on, may be needed.

Ask students to explain why the choice $0 \le y \le 4.5\ HM/W$ was made, and have them find the least positive constant c such that $0 \le y \le c\ HM/W$ might produce a fully functional folding fan (Answer: $c = \pi$). This is a fairly difficult question for most students, but hardly impossible.

Tell students who are studying or have studied multivariate calculus to shade in two rectangles, $\pi/4$ wide by $M/4$ high, on the unfolded strips: one at the top of the strip and one near the middle. Have them use the resulting fan to help explain the r factor in the $dx\,dy \to r\,dr\,d\theta$ change-of-variable formula for multiple integrals.

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A Sequence Related to the Harmonic Series

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Since the harmonic series diverges, the series $\Sigma_{k=n}^{\infty} 1/k$ also diverges for each positive integer n. It follows that there is a least positive integer $a(n)$ such that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{a(n)} > 1,$$

and the resulting sequence $a(n)$ is the focus of our study. Inductions clearly show that

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1} < 1 \qquad \text{and}$$

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n-2} > 1 \qquad \text{for } n \ge 2,$$

so we see that $2n - 1 < a(n) \le 3n - 2$ for all $n \ge 2$.

These bounds on $a(n)$ suggest that the sequence $a(n)/n$ may converge to a limit and, since $\int_n^{ne} 1/x\,dx = 1$, we expect that limit to be the number e.

To verify this limit, first observe from the definition of $a(n)$ that

$$1 < \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{a(n)} \le 1 + \frac{1}{a(n)}.$$

Then, using the left-hand and right-hand side Riemann sums with the integer partition of the interval $[n, a(n)]$, we get the following bounds for $\int_n^{a(n)} 1/x\,dx = \ln[a(n)/n]$:

$$1 - \frac{1}{n} < \frac{1}{n+1} + \cdots + \frac{1}{a(n)} < \ln\frac{a(n)}{n} < \frac{1}{n} + \cdots + \frac{1}{a(n)-1} \leq 1.$$

Thus, since the exponential function is increasing, we have

$$\exp\left(1 - \frac{1}{n}\right) < \frac{a(n)}{n} < e$$

and the pinching theorem yields the result that

$$\lim_{n \to \infty} \frac{a(n)}{n} = e.$$

Apparently, the integer $a(n)$ is located in the interval $I_n = (ne^{1-1/n}, ne)$ whose length is greater than 2 for $n \geq 2$ and increases toward a limiting value of e as n increases. Thus, there are either two or three integers in this interval and the value of $a(n)$ must be one of $[ne]$, $[ne]-1$, and $[ne]-2$. (This can be more readily seen by utilizing the right-hand sum over $[n-1, a(n)]$ to obtain the slightly looser interval $J_n = (ne - e, ne)$.)

Which one is it? A computer check of $2 \leq n \leq 2115$ shows no solutions of $a(n) = [ne] - 2$ and that for most values of n we have $a(n) = [ne] - 1$. Moreover, the exceptional cases in which $a(n) = [ne]$ exhibit an intriguing pattern, as shown in Table 1. Note that most differences between consecutive n-values in the table

Table 1
Values of n for which $a(n) = [ne]$

4	11	18	25	32|	36	43	50	57	64	71|	75	82	89	96	103||
114	121	128	135	143|	146	153	160	167	174||						
185	192	199	206	213|	217	224	231	238	245||						
256	263	270	277	284|	288	295	302	309	316||						
327	334	341	348	355|	359	366	373	380	387||						
398	405	412	419	426|	430	437	444	451	458||						
469	476	483	490	497|	501	508	515	522	529	536|	540	547	554	561	568||
579	586	593	600	607|	611	618	625	632	639||						
650	657	664	671	678|	682	689	696	703	710||						
721	728	735	742	749|	753	760	767	774	781||						
792	799	806	813	820|	824	831	838	845	852||						
863	870	877	884	891|	895	902	909	916	923||						
934	941	948	955	962|	966	973	980	987	994||						
1005	1012	1019	1026	1033||											
1044	1051	1058	1065	1072|	1076	1083	1090	1097	1104||						
1115	1122	1129	1136	1143|	1147	1154	1161	1168	1175||						
1186	1193	1200	1207	1214|	1218	1225	1232	1239	1246||						
1257	1264	1271	1278	1285|	1289	1296	1303	1310	1317||						
1328	1335	1342	1349	1356|	1360	1367	1374	1381	1388||						
1399	1406	1413	1420	1427|	1431	1438	1445	1452	1459||						
1470	1477	1484	1491	1498|	1502	1509	1516	1523	1530	1537|	1541	1548	1555	1562	1569||
1580	1587	1594	1601	1608|	1612	1619	1626	1633	1640||						
1651	1658	1665	1672	1679|	1683	1690	1697	1704	1711||						
1722	1729	1736	1743	1750|	1754	1761	1768	1775	1782||						
1793	1800	1807	1814	1821|	1825	1832	1839	1846	1853||						
1864	1871	1878	1885	1892|	1896	1903	1910	1917	1924||						
1935	1942	1949	1956	1963|	1967	1974	1981	1988	1995||						
2006	2013	2020	2027	2034||											
2045	2052	2059	2066	2073|	2077	2084	2091	2098	2105||						

are 7, syncopated every fifth or sixth entry by a gap of 4 or 11. (Readers seeking to extend this table are advised to employ high-accuracy software.)

Upon further experimentation, one finds that the entries in the table all occur in a larger set of integers n for which $[ne]-2$ is outside the interval I_n; so this does not characterize the property $a(n)=[ne]$, but appears to be a necesasry condition. Is it? Does the pattern indicated by the table persists, or does chaos eventually take over? Is $a(n)=[ne]-2$ impossible?

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A Normal Density Project

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The fact that the normal density function

$$n(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

has no closed-form antiderivative [1], [2] can motivate student projects that are organized around the idea of approximating the cumulative distribution function

$$F(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)dt.$$

Useful projects might be devised by combining portions of the following problems, which range from computer algebra and computer graphing exercises for first-year calculus students to challenging problems in advanced calculus or analysis.

(1) Use a graphing utility to compare the graphs of $f(x)=\exp(-x^2/2)$ and $g_n(x)=(1+x^2/2n)^{-n}$ for $n=1,2,3,\ldots$. It is most impressive to see an animation that lets n cycle through the values 1 to 10 and back again.

(2) Use l'Hôpital's rule to check that

$$\lim_{n\to\infty}\left(1+\frac{x^2}{2n}\right)^{-n}=\exp\left(-\frac{x^2}{2}\right).$$

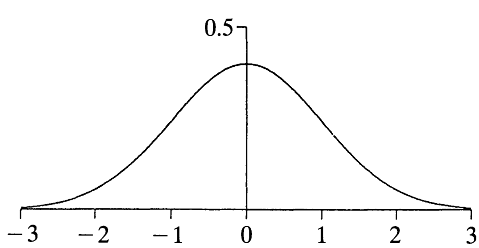

Figure 1

$$y=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

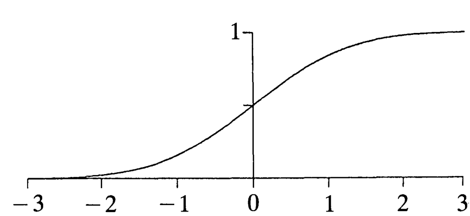

Figure 2

$$y=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)dt$$