It turned out that these were not new discoveries. We'd unexpectedly stumbled upon two old results: the improved trapezoid and Simpson methods are actually cases $n = 2$ and 3 of Gaussian n-point quadrature. (See **[2]**, for example.) The zeros of the Legendre polynomial of degree n lead to an n-point quadrature formula that exactly integrates polynomials through degree $2n - 1$. Our factors $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{3}{5}}$ are zeros of Legendre polynomials of order 2 and 3; correspondingly, our formulas work through degree 3 and 5. Tabulated solutions for $2 \leq n \leq 20$ appear in **[1]** and, through $n = 200$, in **[3]**. Of course, this approach does not fit into a typical beginning course.

Our experience showed that substantial improvements to the trapezoidal and Simpson's methods can be successfully introduced into a course for beginners, using little more than pictures and Maple's **simplify** command.

References

1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, Applied Series No.* 55, National Bureau of Standards, 1964.
2. Richard L. Burden and J. Douglas Faires, *Numerical Analysis*, 5*th ed.*, PWS-Kent Publishing Company, 1993.
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Multiplying and Dividing Polynomials Using Geloxia

Jeff Suzuki (jeffs@bu.edu), College of General Studies, Boston University, Boston, MA 02215

One popular method for multiplying numbers during the Renaissance was that of "geloxia" or the grating **[1**, p. 209**]**. In this system the two numbers to be multiplied were written in an "L" shape above a grid of squares divided by diagonals. In Figure 1 is shown the multiplication of 2375 by 127 to give the product 301625. The entry in each square is the product of the two numbers at the top of the column and the

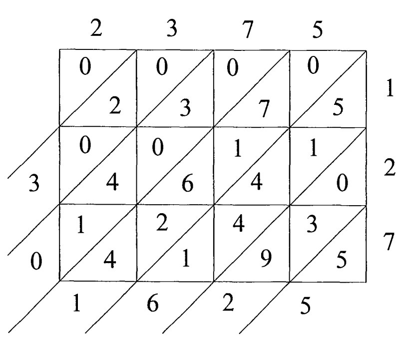

Figure 1

right of the row; for two-digit products the tens digit is written above the diagonal and the units below. The product is found by summing along the diagonals.

The method was popular, though the difficulty of typesetting the grid meant that it would fall out of favor after the invention of printing. It is, of course, equivalent to our method of multiplication, as Figure 2 shows: the product is the sum of all the numbers in the grid.

The same method can be used to multiply two polynomials. For example, here is the multiplication of $3x^2 - 5x + 4$ by $2x - 7$ to give the product $6x^3 - 31x^2 + 43x - 28$: the products of the entries in the rows and columns are entered, the diagonals are added, and the sums indicated at the foot of each diagonal, as shown in Figure 3.

2000	300	70	5	
200000	30000	7000	500	100
40000	6000	1400	100	20
14000	2100	490	35	7

Figure 2

	$3x^2$	$-5x$	4	
	$6x^3$	$-10x^2$	$8x$	$2x$
$6x^3$	$-21x^2$	$35x$	-28	-7
$-31x^2$	$43x$	-28		

Figure 3

There are several advantages to this method. The first is that if students have used the grating method for multiplying integers they will see that the *same* method is used. The usual methods of multiplying integers and polynomials are distinctly different. A second is that the product is very easy to check as all the partial products are obviously written down. A third is that, as all the terms along a diagonal are of the same degree, it is easy to find a term of any desired degree.

The fourth advantage of the grating method is apparent when dealing with quotients. There are several problems, pedagogically, with the normal method of long division. The primary problem is that it is so different from the method of multiplication that the connection between the two operations is often lost.

Using a grating, it is easy to see that they are inverse operations. Let us divide $6x^3 - 31x^2 + 43x - 28$ by $2x - 7$. We write the divisor at the end of the rows (see Figure 4) and the product at the ends of the diagonals. The sum along the first diagonal is $6x^3$ so the only cell in the first diagonal contains $6x^3$. This is the product of $2x$ and the term at the top of the first column, so that term is $3x^2$, as shown in Figure 5.

Now the first column may be filled in. Since $(3x^2)(-7) = -21x^2$, that quantity can be filled in below the $6x^3$. This then determines the entry in the second column of row 1: since the diagonal sum is $-31x^2$, it must be $-10x^2$ (see Figure 6). The term at the top of the second column is now determined: it is $-5x$. The remaining cell in the second column contains $35x$ and this gives $8x$ as the entry in the third column of the first row. Now the last term in the quotient, 4, is determined and the grid may be completely filled in (Figure 7).

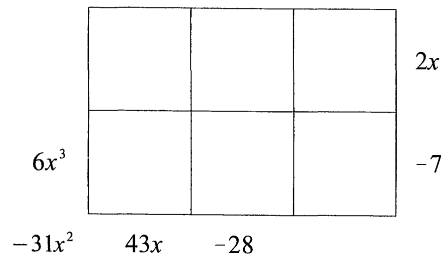

Figure 4

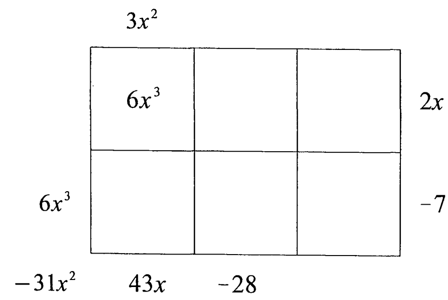

Figure 5

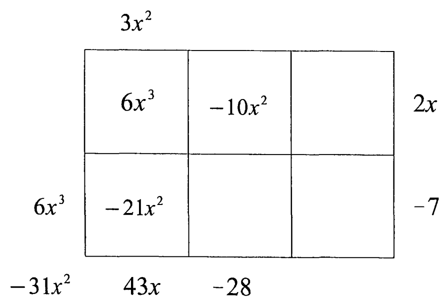

Figure 6

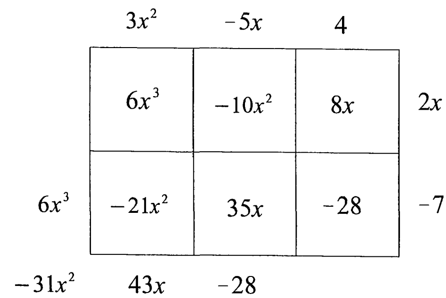

Figure 7

As mentioned before, the connection between multiplication and division is retained: the two use the same grid and, by the end of the process, the multiplication and division grids are identical.

The method handles divisions with remainders as well. Let us divide $4x^3 - 8x^2 + 5x + 11$ by $2x^2 - 5x + 4$. As before a grating grid is written down, with several extra columns, as in Figure 8. It is easy to fill in the quotient and the grid, which then (Figure 9) shows the quotient, $2x + 1$ and the remainder, $2x + 7$.

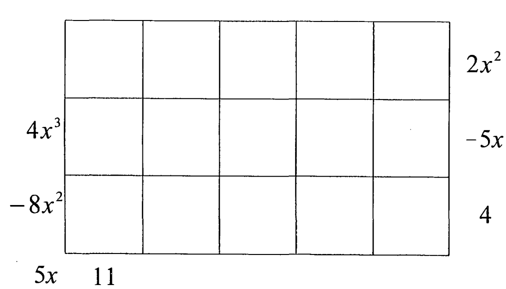

Figure 8

	$2x$	1				
	$4x^3$	$2x^2$	$2x$	7		$2x^2$
$4x^3$	$-10x^2$	$-5x$				$-5x$
$-8x^2$	$8x$	4				4
$5x$	11					

Figure 9

Of course, the grid must be drawn correctly to allow for "missing" entries. Figure 10 is the grid for $(x^4 + 7x^2 + 8)/(x^2 + 2) = x^2 + 5 - 2/(x^2 + 2)$.

	x^2	$0x$	5			
	x^4	$0x^3$	$5x^2$	$0x$	-2	x^2
x^4	$0x^3$	$0x^2$	$0x$			$0x$
$0x^3$	$2x^2$	$0x$	10			2
$7x^2$	$0x$	8				

Figure 10

It is even possible to use the grid method to introduce infinite series. We will leave it to the reader to divide 1 by $-x + 1$ to get $1 + x + x^2 + x^3 + \cdots$ with the hint that the grating should extend indefinitely to the left.

Reference

1. Frank J. Swetz. *Capitalism and Arithmetic*, Open Court, 1987.