Introducing Hyperbolicity via Piecewise Euclidean Complexes

Jessica Benashaski, John Meier (meierj@lafayette.edu), Kevin O'Brien, Paige Reinheimer, and Margaret Skarbek, Lafayette College, Easton, PA 18042

Many topics in a sophomore level geometry class can be presented in a very tangible fashion. For example, spherical geometry can be introduced by bringing string, stick-um, and a few large rubber balls to class. Students can use these manipulatives to experiment with the angle sums of spherical triangles, building intuition that leads to a proof that the area of a spherical triangle is directly related to its angular excess. (For other examples of exploratory activities in elementary geometry see [2] and [3].)

While spherical geometry is relatively easy to introduce through such in-class exercises, hyperbolic geometry is much more difficult. You can bring potato chips to class to indicate the local geometry of the hyperbolic plane, but it isn't so easy to use them to explore geodesics and triangles. Because they are small and brittle, it's hard to construct geodesics on potato chips. Also, it's not clear how to measure angles on such surfaces. One can estimate angles between geodesics on a sphere by placing a sheet of paper (= tangent plane) on the sphere, and then measuring the angle between the tangent lines to the geodesics. You can't do this with a physical model of the hyperbolic plane since the hyperbolic plane is negatively curved, and any tangent plane you make will cut through the surface.

If you are willing to let go of the local hyperbolic structure in favor of a model that mimics hyperbolic geometry on a large scale, then *Thurston paper* does the trick. Thurston paper is briefly described in [5] and [6] and is the subject of exercise 2.1.4 in Thurston's book [4]. None of these sources discuss the topic we present here, that the geometry of geodesics in Thurston paper nicely approximates the geometry of geodesics in the hyperbolic plane. That this approach to hyperbolic geometry is accessible to students is evidenced by the fact that four of the five authors wrote the first draft of this paper as part of a student research project!

The Euclidean plane can be subdivided into equilateral triangles, where six triangles are joined at every vertex. Thurston paper is constructed by joining together seven equilateral triangles at every vertex. The addition of the extra triangle causes the system to bend and twist, instead of lying flat like the Euclidean plane.

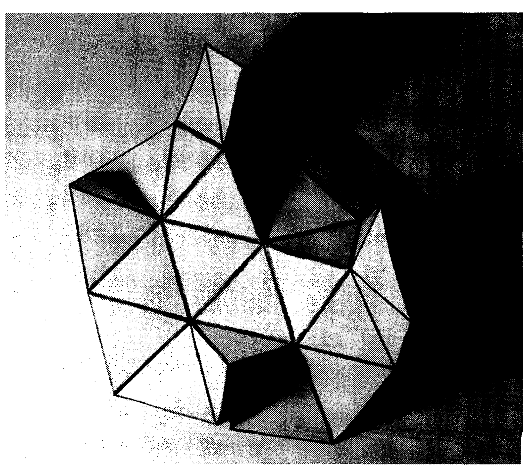

Figure 1. A photo of Thurston paper.

It's a lot of fun to make Thurston paper. With the aid of many hands, a large copy of Thurston paper can be made in a short amount of time. Students take great delight in having it wrap around their arms, or using it as a hat. If you've never made your own copy, we suggest you make one before reading further!

There are bits—actually entire strips—of the Euclidean plane hidden in Thurston paper. In the Euclidean plane, constructed from equilateral triangles, there are *corridors* consisting of an infinitely long band formed by triangles alternately pointing up and down. These corridors also exist in Thurston paper.

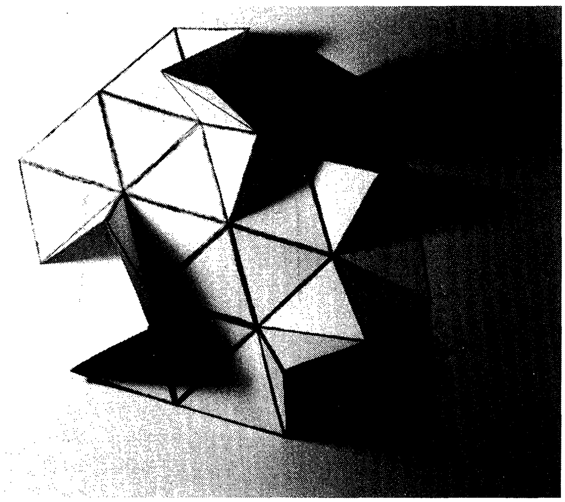

Figure 2. A Euclidean corridor in Thurston paper.

(In later figures we'll distort the metric on Thurston paper so that we can represent it as though it were "flat" on the page. This has the added benefit of making Thurston paper appear similar to the unit disk model of $\mathbb{H}^2$.)

A triangle in Thurston paper consists of three distinct points joined by geodesics (= locally length minimizing paths). In order to construct triangles in Thurston paper students need to discover simple rules for finding geodesics. To develop some intuition, students can experiment with the idea of geodesic in much the same manner as they would with a sphere. On a sphere one finds geodesics by pulling a bit of string tight between two points. Similarly, to find the geodesic between a pair of points in Thurston paper you pinch the two points between your fingers, and carefully pull the paper tight.

Students quickly discover that if two points are contained in a Euclidean corridor, then the straight line running between them in the Euclidean corridor is a geodesic in Thurston paper. So in some sense Thurston paper fails as a model of hyperbolic geometry; in Thurston paper there are very long and skinny triangles, embedded in Euclidean corridors, that look exactly like Euclidean triangles. However, once you step outside of the Euclidean corridors, the geometry of Thurston paper is decidedly hyperbolic in that the sum of the angles of a triangle can be less than 180°, as we show in the two examples below.

Take two line segments with common endpoint x that do not intersect any edge or vertex, except possibly at x itself. These two line segments divide a small metric neighborhood of x into two pieces, each piece being composed of Euclidean sectors. One can define the *angle* formed by the line segments (with respect to a chosen side) to be the sum of the Euclidean angles of the sectors. One can show (and it makes a good exercise for students) that a path p is a geodesic if for any point x on the path,

- If x is in the interior of a triangle or an edge, then there is a neighborhood of x in which p is a Euclidean geodesic. [This makes sense if x is interior to a triangle since our triangles are Euclidean. It also makes sense when x is interior to an edge, because the union of the two triangles adjoining an edge is isometric to a Euclidean parallelogram.]
- If x is a vertex, then the angles between the path into and the path out of x, on both sides, are $\geq 180°$.

We refer to the second condition as the 180°-*criterion*. Using the criteria above one sees that the line joining X and Y in Figure 3 is a geodesic.

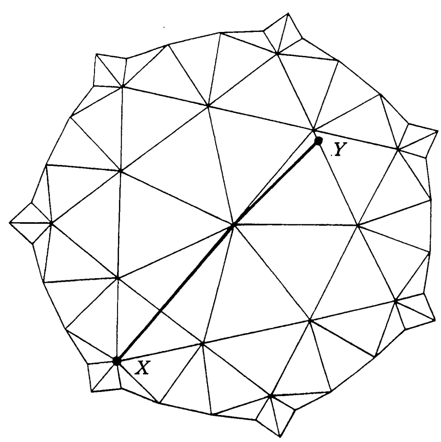

Figure 3. A geodesic in Thurston paper.

Using an Euler characteristic argument one can prove that geodesics in Thurston paper are unique, but such an argument is beyond the sophomore level. For details on geodesics in piecewise Euclidean complexes see **[1]**.

We illustrate the hyperbolicity of Thurston paper with two triangles (sketched in

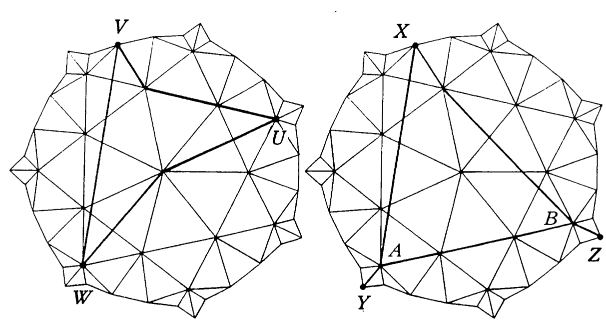

Figure 4. Two triangles in Thurston paper.

Fig. 4) in which the sum of the interior angles is strictly less than 180°. Remember, these flattened pictures are distorted, so the paths drawn are only *indications* of where the actual geodesics lie. If you have a copy of Thurston paper handy, we suggest you find three points in the same relative position and use the pull-the-paper-tight technique to locate the geodesics.

Triangle *UVW*. Because there is a Euclidean corridor containing the points V and W, the geodesic between them is contained in this corridor. The path from U to V runs along the edges of Thurston paper, and it is easy to check that the 180°-criterion is satisfied at each vertex. Finally, the path joining U to W consists of four altitudes from four Euclidean triangles. The 180°-criterion holds at the midpoint of this path because the angle sum on one side is 180° while it's 240° on the other.

The angle $\angle VUW$ is 30° since $\overrightarrow{UV}$ runs along one side of an equilateral triangle and $\overrightarrow{UW}$ starts along the altitude of the same triangle. The other two angles require a bit more ingenuity to measure. Because $\overrightarrow{VW}$ is contained in a Euclidean corridor, we can accurately represent it as sitting in a thin strip of the Euclidean plane, ignoring the ambient Thurston paper. So in Figure 5 we emphasize the Euclidean corridor containing V and W, as well as the geodesics *near* V and W, and we essentially ignore the portion of Thurston paper near U. An initial segment of the geodesic $\overrightarrow{VU}$ starts at the top of the corridor, and an initial segment of $\overrightarrow{WU}$ forms an altitude of a triangle in this corridor. By extending the corridor so that $\overrightarrow{UW}$ is the hypotenuse of a right-angled Euclidean triangle, one sees that $\angle UVW = \tan^{-1}(\sqrt{3}/7)$, $\angle VWU = [\tan^{-1}(\sqrt{3}/3)]/2$, and most importantly, $\angle UVW + \angle VWU = 30°$. So the sum of the interior angles of triangle UVW is 60°.

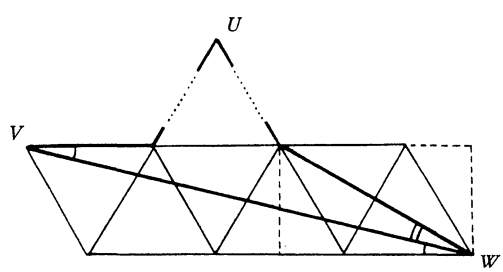

Figure 5. The Euclidean corridor containing V and W.

Triangle *XYZ*. Using similar methods to those in the previous example one can check that $\angle YXZ = \tan^{-1}(\sqrt{3}/7)$. The angles $\angle XYZ$ and $\angle XZY$ introduce one further subtlety. Since $\overrightarrow{YA}$ is contained in both $\overrightarrow{YX}$ and $\overrightarrow{YZ}$, the "angle" formed by these two distinct geodesics is zero! The same sort of bifurcation occurs with the geodesics $\overrightarrow{ZY}$ and $\overrightarrow{ZX}$. If one agrees that these angles should be considered to be zero, then the sum of the interior angles of this triangle is about 14°. On the other hand, it's not unreasonable to measure $\angle XYZ$ at the point of bifurcation, in which case techniques like those discussed in the previous example show that the angle sum is a bit over 79°.

The geometry of geodesics is a nice starting point for further student investigations. One can find experimental evidence that larger triangles have smaller angle sums in Thurston paper, although it would be impossible to get any precise correlation. As in $\mathbb{H}^2$ there are lines $\mathscr{L}$ and points p where there are infinitely many distinct lines through p that do not intersect $\mathscr{L}$. One can also explore isoperimetric inequalities, showing that (roughly speaking) the area enclosed by a circle of radius r in Thurston paper grows exponentially as r increases. The particular exploration you might assign would depend on the goals of your course, but in any event, after working with Thurston paper your students should have sufficient insight and motivation to begin a more rigorous discussion of hyperbolic geometry.

References

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t-Probabilities as Finite Sums

Neil Eklund (eklund@centre.edu), Centre College, Danville, KY 40422-1394

A text **[1]** has an exercise to derive the *t*-probability formula,

$$P[t \geq R, df = N] = \frac{\Gamma((N+1)/2)}{\sqrt{N\pi}\,\Gamma(N/2)} \int_R^\infty (1 + x^2/N)^{-(N+1)/2}\, dx \equiv P_N,$$

where *df* denotes the number of degrees of freedom. The purpose of this note is to show how this expression may be written as a finite sum and thus may be evaluated by writing a program on any programmable calculator.

Substitute $x = N^{1/2} \tan\theta$ and define φ by $R = N^{1/2} \tan\varphi$. Then P_N becomes

$$P_N = \frac{\Gamma((N+1)/2)}{\sqrt{\pi}\,\Gamma(N/2)} \int_\varphi^{\pi/2} \cos^{N-1}\theta\, d\theta.$$

Since φ depends on N, define for $K \geq 1$

$$Q_K = Q_K(N) \equiv \frac{\Gamma((K+1)/2)}{\sqrt{\pi}\,\Gamma(K/2)} \int_\varphi^{\pi/2} \cos^{K-1}\theta\, d\theta$$

and note that $P_N = Q_N(N)$. In order to compute P_N we use the reduction formula

$$\int \cos^m\theta\, d\theta = \frac{1}{m}(\sin\theta)(\cos^{m-1}\theta) + \frac{m-1}{m}\int \cos^{m-2}\theta\, d\theta.$$

Since Q_1 and Q_2 can be computed directly, we have for $K \geq 3$

$$Q_K = -\frac{\Gamma((K+1)/2)}{\sqrt{\pi}\,\Gamma(K/2)}\frac{1}{K-1}(\sin\varphi)(\cos^{K-2}\varphi) + \frac{\Gamma((K+1)/2)}{\sqrt{\pi}\,\Gamma(K/2)}\frac{K-2}{K-1}\int_\varphi^{\pi/2} \cos^{K-3}\theta\, d\theta \tag{1}$$