Old Calculus Chestnuts: Roast, or Light a Fire?

Margaret Cibes, University of Hartford, West Hartford, CT 06117

George Polya warned us a long time ago about the necessity to recognize the many opportunities presented by and among problems:

> Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. ...A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted.
>
> One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else. We have a natural opportunity to investigate the connections of a problem when looking back at its solution. [**1**]

Year after year calculus classes work on optimization problems involving geometric figures. Are we teachers, like most textbook authors, limiting our expectations of students to procedurally correct write-ups of case-specific results with answers that match those in the back of the book? Or are we encouraging our students to reflect upon their answers, discover patterns in them, and connect these patterns to those in other problems? I offer here three familiar problems that will reward extra reflection.

First is the classic problem of *maximizing the area of a rectangular field* that can be enclosed on 3 or 4 sides and/or subdivided with a given length of fence. Consider the ordinary case in which fence lengths for opposite sides, if they both exist, are equal and any fence length that subdivides the interior is parallel to a side and equal in length to it. (The existence of a maximum area, of course, depends upon at least one "vertical" side and one "horizontal" side being fenced. Otherwise, there is no upper limit to the area that could be fenced in.)

One pattern that becomes evident from looking at one case and then removing or adding fence lengths for other cases is that the maximum area is achieved when the sum of the lengths of the "vertical" pieces of fence is equal to the sum of the lengths of the "horizontal" pieces of fence. Each sum equals one-half of the total fence length, and so in a sense every maximizing rectangle has one of the regularities of a square, a striking result. But there is more. The fence problem can be connected to our second problem: *maximizing the volume of a square-base box*, with or without a "lid," given a total surface area. One might conjecture an analogous regularity in the relationship between the total surface area of the "horizontal" square end(s) and that of the other four "vertical" sides. One might further investigate this relationship when the box is without one or more "vertical" sides.

The first table below gives the results after removal of 0, 1, or 2 faces, as well as the general case of covering m ends and n sides.

In order to relate our optimal configurations to the cube, we have expressed values in terms of the length of the edge of an ordinary cube with given surface area A, that is, $x = \sqrt{A/6}$. The term "end" refers to one of the two "horizontal" square faces of the box; "side" refers to one of the four "vertical" rectangular faces; and "Total Area" refers to the total area of the faces that have not been removed but are to be covered with the given material.

Optimization of Volume of Box					
Face(s) Removed	Critical Values: End Length	Critical Values: Side Height	Total Area of End(s)	Total Area of Side(s)	Maximum Volume
None	x	x	$2\,x^2$	$4\,x^2$	x^3
One End	$\sqrt{2}\,x$	$\frac{1}{\sqrt{2}}\,x$	$2\,x^2$	$4\,x^2$	$\sqrt{2}\,x^3$
One Side	x	$\frac{4}{3}\,x$	$2\,x^2$	$4\,x^2$	$\frac{4}{3}\,x^3$
Two Ends	none	none			none
One End & One Side	$\sqrt{2}\,x$	$\frac{4}{3}\frac{1}{\sqrt{2}}\,x$	$2\,x^2$	$4\,x^2$	$\frac{4}{3}\sqrt{2}\,x^3$
Two Sides	x	$2\,x$	$2\,x^2$	$4\,x^2$	$2\,x^3$
GENERAL CASE $m, n \neq 0$	$\sqrt{\frac{2}{m}}\,x$	$\frac{4}{n}\sqrt{\frac{m}{2}}\,x$	$2\,x^2$	$4\,x^2$	$\frac{4}{n}\sqrt{\frac{2}{m}}\,x^3$

The conjecture about surface areas is substantiated. For an optimal configuration, we find that the ratio of total surface area of the covered ends to the total surface area of the covered sides is 1 to 2, just as for the cube!

Where could the calculus student go from here? The next connection might be to a third classic problem: *maximizing the volume of a cylinder* with a given surface area. The basic results are shown in the table below.

In order to connect this cylinder problem to the preceding box problem, we again use $x = \sqrt{A/6}$ as our unit of length.

Optimization of Volume of Cylinder					
Faces Removed	Critical Values: Radius	Critical Values: Height	Total Area of Ends	Total Area of Side	Maximum Volume
None	$\frac{1}{\sqrt{\pi}}\,x$	$\frac{2}{\sqrt{\pi}}\,x$	$2\,x^2$	$4\,x^2$	$\frac{2}{\sqrt{\pi}}\,x^3$
One End	$\sqrt{\frac{2}{\pi}}\,x$	$\sqrt{\frac{2}{\pi}}\,x$	$2\,x^2$	$4\,x^2$	$2\sqrt{\frac{2}{\pi}}\,x^3$

Here again in each case the surface area ratio of ends to side is 1 to 2.

One might inquire further about the relative sizes of the boxes and cylinders for a given surface area. Scale drawings or models might prove very fruitful here. One might also vary the box or cylinder by subdividing the interior with pieces of the given material to provide "shelves." See the general case in the table for the box.

Inquiries such as those described above keep mathematics alive as an ongoing, creative process in our students and not dead as a list of theorems, problems, and solutions in textbooks. Leon Henkin described it aptly:

> One of the big misapprehensions about mathematics that we perpetrate in our classrooms is that the teacher always seems to know the answer to any

problem that is discussed. This gives students the idea that there is a book somewhere with all the right answers to all of the interesting questions, and that teachers know those answers. And if one could get hold of the book, one would have everything settled. That's so unlike the true nature of mathematics. [**2**]

References

1. G. Polya, *How To Solve It*, Princeton University Press, Princeton, NJ, 1946, p. 14.
2. L. A. Steen and D. J. Albers, eds., *Teaching Teachers, Teaching Students*, Birkhauser, Boston, 1981, p. 89.

Least Squares and Quadric Surfaces

Donald Teets, South Dakota School of Mines and Technology, Rapid City, SD 57701-3995

We have all seen fairly difficult problems in college algebra texts that would be easy "if only the student knew calculus." But have you ever seen a difficult problem in a calculus text that would be easier if only the student *didn't* use calculus? The purpose of this note is to describe such a problem.

The derivation of the least squares regression line $f(x) = ax + b$ for the n data points $(x_1, y_1), \ldots, (x_n, y_n)$ (where $n \geq 2$ and the x_i's are not all the same) is commonly presented as an application of minimizing a multivariable function [**1**], [**2**], [**3**]. The standard approach to this problem is to minimize the sum of the squared errors

$$s(a,b) = \sum_{i=1}^{n} (ax_i + b - y_i)^2 \tag{1}$$

by setting the partial derivatives $s_a(a,b)$ and $s_b(a,b)$ equal to zero and solving for a and b, obtaining

$$a = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}, \qquad b = \frac{1}{n}\left(\sum y_i - a\sum x_i\right). \tag{2}$$

(Here and in the remainder of this note, Σ means $\Sigma_{i=1}^{n}$.) This, of course, is insufficient to show that these values minimize $s(a,b)$; thus we find the following exercise in [**3**]: "Use the Second-Partials Test to verify that the formulas given for a and b yield a minimum." (The reader is invited to try this exercise before reading further!)

For the function $s(a,b)$ to have a minimum at the point (a,b) given in (2), the second-partial test requires that

$$s_{aa}(a,b)s_{bb}(a,b) - [s_{ab}(a,b)]^2 > 0$$

at that point. Upon computing the derivatives, this reduces to

$$n\sum x_i^2 - \left(\sum x_i\right)^2 > 0. \tag{3}$$