CLASSROOM CAPSULES

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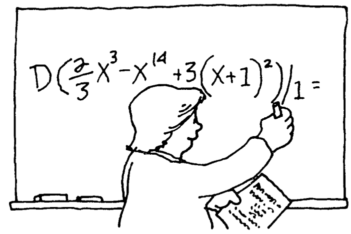

A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to Frank Flanigan.

The Relationship Between Hyperbolic and Exponential Functions—Revisited

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In the standard calculus course, hyperbolic functions are defined in terms of exponential functions, i.e., $\sinh t = \frac{1}{2}(e^t - e^{-t})$ and $\cosh t = \frac{1}{2}(e^t + e^{-t})$. Later certain identities are verified, including $\cosh^2 t - \sinh^2 t = 1$; from which it follows that the point $(\cosh t, \sinh t)$ lies on the right-hand branch of the hyperbola $x^2 - y^2 = 1$. Thus the name "hyperbolic" is belatedly justified. But this observation provides no *motivation* for the choice of these particular combinations of exponential functions in defining $\cosh t$ and $\sinh t$.

To remedy this, we can begin with $(\cosh t, \sinh t)$ as a point on the unit hyperbola $x^2 - y^2 = 1$ in a fashion analogous to the definition of the circular functions $(\cos t, \sin t)$ as the coordinates of a point on the unit circle $x^2 + y^2 = 1$. Just as $t/2$ measures the signed area of the circular sector swept out as a radius to $(1, 0)$ rotates to $(\cos t, \sin t)$, let us *define* $(\cosh t, \sinh t)$ to be the coordinates of a point on the right-hand branch of $x^2 - y^2 = 1$ so that the "radius" has again swept out a signed area of $t/2$ (Figures 1 and 2).

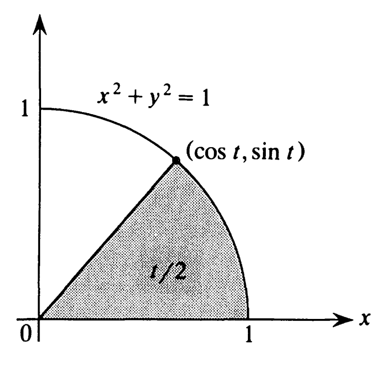

Figure 1

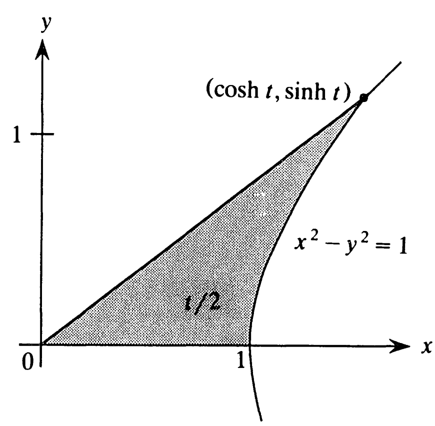

Figure 2

An earlier capsule [CMJ 19 (January 1988) 54–56] demonstrated how a 45° rotation of axes could be used with this definition to derive the familiar formulas for the hyperbolic functions in terms of exponential functions. In this capsule, we replace rotation of axes by integration in polar coordinates (and some elementary identities with circular functions) to accomplish the same end somewhat more simply.

To simplify the notation, let $(u,v)=(\cosh t, \sinh t)$ and set $\alpha = \arctan(v/u)$, $\alpha \in (-\pi/4, \pi/4)$. The shaded region in Figure 2 is bounded by $r^2\cos 2\theta = 1$ (the polar representation of the hyperbola $x^2 - y^2 = 1$) and the rays $\theta = 0$ and $\theta = \alpha$. Hence its (signed) area $t/2$ is equal to $\frac{1}{2}\int_0^\alpha r^2\,d\theta = \frac{1}{2}\int_0^\alpha \sec 2\theta\,d\theta$. Thus

$$\begin{aligned} t &= \tfrac{1}{2}\ln|\sec 2\alpha + \tan 2\alpha| \\ &= \tfrac{1}{2}\ln\left|\frac{1+\sin 2\alpha}{\cos 2\alpha}\right|. \end{aligned}$$

But $\dfrac{1+\sin 2\alpha}{\cos 2\alpha} > 0$ since $\alpha \in (-\pi/4, \pi/4)$, and

$$\begin{aligned} \frac{1+\sin 2\alpha}{\cos 2\alpha} &= \frac{(\cos\alpha + \sin\alpha)^2}{\cos^2\alpha - \sin^2\alpha} = \frac{\cos\alpha + \sin\alpha}{\cos\alpha - \sin\alpha} \\ &= \frac{1+\tan\alpha}{1-\tan\alpha} = \frac{1+\dfrac{v}{u}}{1-\dfrac{v}{u}} = \frac{u+v}{u-v}, \end{aligned}$$

so that

$$t = \frac{1}{2}\ln\left(\frac{u+v}{u-v}\right).$$

But recall that $u^2 - v^2 = 1$, so that $\dfrac{u+v}{u-v} = (u+v)^2 = (u-v)^{-2}$ (with both $u-v$ and $u+v$ positive), and thus we can construct the following pairs of equations

$$\begin{cases} t = \frac{1}{2}\ln(u+v)^2 \\ t = \frac{1}{2}\ln(u-v)^{-2} \end{cases} \Rightarrow \begin{cases} t = \ln(u+v) \\ -t = \ln(u-v) \end{cases} \Rightarrow \begin{cases} e^t = u+v \\ e^{-t} = u-v \end{cases}.$$

Solving the last pair for u and v gives the familiar expressions for the hyperbolic cosine and sine.

At this point the reader may ask why we don't just integrate directly in rectangular coordinates to find the area of the shaded region in Figure 2. Of course, this approach can be taken, but it is rather more complicated. It leads to the integral $\int(x^2-1)^{1/2}\,dx$ which, after a trigonometric substitution, becomes $\int \sec^3\theta\,d\theta$, requiring integration by parts. We leave the details to the reader.

We close by noting that an approach to the hyperbolic functions involving motivation along the lines discussed here will generally require delaying their introduction in the typical calculus course. An examination of several popular calculus texts reveals that polar coordinates (as well as rotation of axes or integration by parts) follows the hyperbolic functions—as most authors present these functions as an application of exponential functions rather than as analogous to circular functions.