are 7, syncopated every fifth or sixth entry by a gap of 4 or 11. (Readers seeking to extend this table are advised to employ high-accuracy software.)

Upon further experimentation, one finds that the entries in the table all occur in a larger set of integers n for which $[ne]-2$ is outside the interval I_n; so this does not characterize the property $a(n)=[ne]$, but appears to be a necesasry condition. Is it? Does the pattern indicated by the table persists, or does chaos eventually take over? Is $a(n)=[ne]-2$ impossible?

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A Normal Density Project

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The fact that the normal density function

$$n(x)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

has no closed-form antiderivative [1], [2] can motivate student projects that are organized around the idea of approximating the cumulative distribution function

$$F(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)dt.$$

Useful projects might be devised by combining portions of the following problems, which range from computer algebra and computer graphing exercises for first-year calculus students to challenging problems in advanced calculus or analysis.

(1) Use a graphing utility to compare the graphs of $f(x)=\exp(-x^2/2)$ and $g_n(x)=(1+x^2/2n)^{-n}$ for $n=1,2,3,\dots$. It is most impressive to see an animation that lets n cycle through the values 1 to 10 and back again.

(2) Use l'Hôpital's rule to check that

$$\lim_{n\to\infty}\left(1+\frac{x^2}{2n}\right)^{-n}=\exp\left(-\frac{x^2}{2}\right).$$

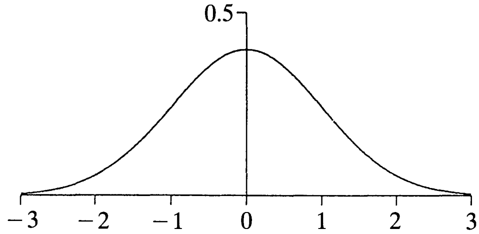

Figure 1

$$y=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$$

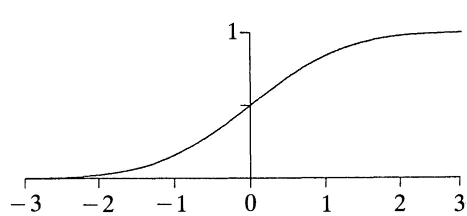

Figure 2

$$y=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2}{2}\right)dt$$

(3) Show that the convergence in (2) is uniform on any interval $[a, b]$ because g_n is a decreasing sequence of continuous functions. Conclude that

$$\int_a^b e^{-x^2/2}\,dx = \lim_{n\to\infty} \int_a^b \left(1 + \frac{x^2}{2n}\right)^{-n} dx,$$

and this gives us a way of approximating the cumulative distribution function if we can evaluate the integral on the right.

(4) Use integration by parts to derive the following reduction formula for the integral of $g_n(x)$:

$$\int \left(1 + \frac{x^2}{k}\right)^{-n} dx = \frac{x\left(1 + \frac{x^2}{k}\right)^{-n+1}}{2n-2} + \frac{2n-3}{2n-2}\int \left(1 + \frac{x^2}{k}\right)^{-n+1} dx$$

for $n > 1$ and $k \neq 0$. use a computer algebra utility to evaluate

$$\int \left(1 + \frac{x^2}{2n}\right)^{-n} dx$$

with $n = 1$, 2, 3, and 4. In particular, verify that

$$\int \left(1 + \frac{x^2}{8}\right)^{-4} dx = \frac{5\sqrt{2}}{8}\arctan\left(\frac{x}{\sqrt{8}}\right) + \frac{x(15x^4 + 320x^2 + 2112)}{6(x^2+8)^3} + C.$$

(5) Use the reduction formula of (4) to demonstrate that

$$\int \left(1 + \frac{x^2}{2n}\right)^{-n} dx = A\arctan\left(\frac{x}{\sqrt{2n}}\right) + \frac{N(x)}{D(x)} + C,$$

where $N(x)$ and $D(x)$ are polynomials and A is a constant that depends on n. Also, $N(0) = 0$, the degree of $N(x)$ is 1 less than the degree of $D(x)$, and N is an odd function while D is even. What are the horizontal asymptotes of such a function?

(6) Use a computer graphing utility to compare the graphs of

$$y = \int_0^x e^{-t^2/2}\,dt \qquad \text{and} \qquad y = \int_0^x \left(1 + \frac{t^2}{2n}\right)^{-n} dt$$

taking $n = 1, 2, 3, \ldots$. The first integral must be evaluated numerically, but the second should be evaluated as in (4).

Projects such as this illustrate how the classically trained mathematician can join forces with the contemporary, computer-literate mathematics student to advance the cause of mathematical insight.

References

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2. E. A. Marchisotto and G.-A. Zakeri, An invitation to integration in finite terms, *College Mathematics Journal* 25:4 (1994) 295–308.

Hyperbolic Functions and Proper Time in Relativity

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The hyperbolic functions have a connection with time dilation in relativity theory that can enrich discussions of these functions and the arc length formula in a calculus class.

It is well known that any point P on the unit circle $x^2+y^2=1$ has coordinates $(\cos s, \sin s)$, for a real number s that may be taken to be in $[0, 2\pi)$. The parameter s has several geometric interpretations (see Figure 1a):

(i) s is twice the area of the circular arc AOP, and
(ii) s is the length of the circular arc AP; that is, the radian measure of the central angle AOP.

For the *unit hyperbola* $x^2-y^2=1$, what analogous statements can be made?

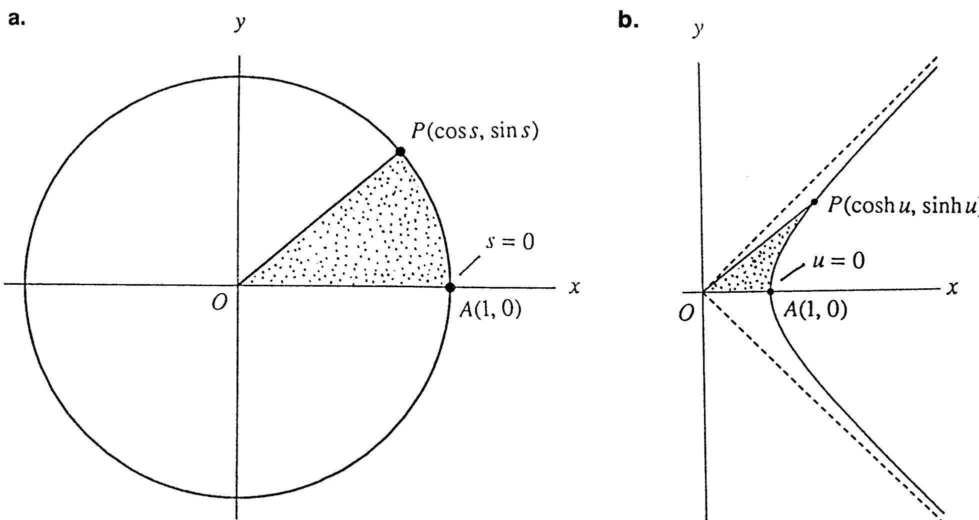

Figure 1

Any point P on the right branch of the unit hyperbola has coordinates $(\cosh u, \sinh u)$, as in Figure 1b, and it is well known **[1]** that in analogy with (i) above, u is twice the area of the hyperbolic sector AOP. Can u also be interpreted as an arc length, in analogy with (ii)?

The initial answer to this is "No"; an easy estimate shows that the length of the hyperbolic arc AP increases approximately exponentially with u. However, this is relative to the usual Pythagorean metric (infinitesimal arc length formula) $ds^2 = dx^2 + dy^2$ of Euclidean space. If instead the arc length is calculated relative to the Lorentz metric (sometimes called an *indefinite* metric or *pseudometric*) $du^2 = -dx^2 + dy^2$, then u is indeed the length of arc AP. When the usual arc length integral is