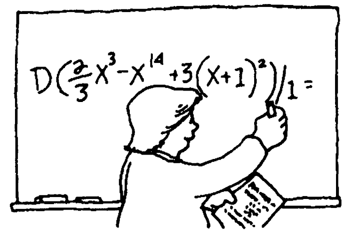
EDITOR

Warren Page
30 Amberson Ave.
Yonkers, NY 10705



A Classroom Capsule is a short article that contains a new insight on a topic taught in the earlier years of undergraduate mathematics. Please submit manuscripts prepared according to the guidelines on the inside front cover to the Editor-elect, Warren Page, 30 Amberson Ave., Yonkers, NY 10705-3613.

What's Harmonic about the Harmonic Series?

David E. Kullman (kullmade@muohio.edu), Miami University, Oxford, OH 45056

Why is $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ called the harmonic series? A simple answer would be that each term of the series, after the first, is the harmonic mean of its two nearest neighbors. This response, however, is likely to raise more questions than it answers: What is the harmonic mean? Where and when did it originate? Is there any connection to musical harmony?

The Harmonic Mean

The arithmetic mean of a and b, $A = \frac{a+b}{2}$, should be familiar to all mathematics students, and the geometric mean, $G = \sqrt{ab}$, is almost as well known. The harmonic mean, however, has been called "the neglected mean" [**4**], [**8**] because it appears so infrequently in core mathematics courses. The *harmonic mean* of two numbers a and b is defined by the formula $H = \frac{2ab}{a+b}$ or, equivalently, $\frac{2}{H} = \frac{1}{a} + \frac{1}{b}$. It follows that, for $n > 1$, $\frac{1}{n}$ is the harmonic mean of $\frac{1}{n-1}$ and $\frac{1}{n+1}$. More generally, if x, $x + y$, and $x + 2y$ are in arithmetic progression, then their reciprocals, $a = \frac{1}{x}$, $H = \frac{1}{x+y}$, and $b = \frac{1}{x+2y}$ are said to be in *harmonic progression.*

The origins of the harmonic mean can be traced back to the ancient Pythagoreans. According to the commentator Proclus (410–485), Pythagoras himself (6th century BC) learned about three means—arithmetic, geometric, and subcontrary (later called harmonic)—while visiting Mesopotamia [**1**]. According to this ancient tradition, Pythagoras also knew a "golden proportion" relating these means. Given two numbers, the first is to their arithmetic mean as their harmonic mean is to the second. That is, $\frac{a}{(a+b)/2} = \frac{2ab/(a+b)}{b}$. Another way to express this is that the harmonic mean is the ratio of the square of the geometric mean to the arithmetic mean ($H = G^2/A$).

"Arithmetic" in the time of the ancient Greeks consisted of the study of properties of whole numbers and rational numbers. The arithmetic mean of two rational numbers must also be rational but, for most pairs of integers, the geometric mean is irrational, and the Greeks relegated such magnitudes to the realm of geometry. These facts may help to explain the names of two of our three means. Later, we'll examine a possible explanation for the name of the harmonic mean.

In passing, we note that the term "harmonic range of points" is also derived from the harmonic mean. Collinear points A, B, C, D are said to form a *harmonic range* if and only if their cross ratio, $(AB, CD) = (\overline{AC}/\overline{CB})/(\overline{AD}/\overline{DB})$, is -1. It can be shown [**5**] that this is equivalent to $2/\overline{AB} = 1/\overline{AC} + 1/\overline{AD}$. That is, the length of AB is the harmonic mean of the lengths of AC and AD.

A Musical Connection

Pythagoras (or one of his disciples) is also credited with giving Western Europe its first theory of music, based on musical intervals that can be expressed in terms of numerical ratios of small whole numbers. Among these are the *octave* (2:1), the *fifth* (3:2), and the *fourth* (4:3). Here the numbers 1, 2, 3, and 4 are proportional to the frequencies of the tones, a larger number corresponding to a higher pitch. The tone corresponding to 1 is called the *tonic*. The names fourth, fifth, and octave come from the ordering of tones on an 8-tone diatonic scale.

The ratio 2:1 for an octave may also be expressed as 12:6, so that the tonic has relative frequency 6. Suppose that the ancient Pythagoreans wanted to introduce a new tone, roughly midway between the tonic and the octave. The geometric mean of 6 and 12 is irrational, so a musical tone based on it would have been unacceptable. The arithmetic mean of 6 and 12 is 9, leading to the ratios 9:6 = 3:2 and 12:9 = 4:3. In such a scale the new tone (corresponding to a relative frequency of 9) would be a fifth above the tonic and the octave would be a fourth above the new tone.

On the other hand, the harmonic mean of 6 and 12 is 8, producing ratios 8:6 = 4:3 and 12:8 = 3:2. Now the new tone is a fourth above the tonic, with the octave a fifth above that. A surviving fragment of the work of Archytas of Tarentum (ca. 350 BC) states, "There are three means in music: one is the arithmetic, the second is the geometric, and the third is the subcontrary, which they call harmonic" [**7**]. The term subcontrary may refer to the fact that a tone based on this mean reverses the order of the two fundamental musical intervals in a scale.

It is believed that Archytas or one of his contemporaries gave the name "harmonic" to the subcontrary mean because it produced a division of the octave in which the middle tone stood in the most harmonious relationship to the tonic and the octave. Even today, many musicians prefer a scale in which a fourth is followed by a fifth. (A more complete discussion of Greek musical theory may be found in [**2**] and [**9**].)

Divergence of the Harmonic Series

The earliest recorded proof that the harmonic series diverges is found in a treatise by the 14th century Parisian scholar, Nicole Oresme. In the third proposition of his work, *Questions on the geometry of Euclid* (ca. 1350) Oresme asserted that, "It is possible that an addition could be made, though not proportionally, to any quantity by ratios of lesser inequality, and yet the whole would become infinite" [**6**]. His example consists of adding to a one-foot quantity one-half of a foot, then one-third of a foot, then one-fourth and one-fifth "and so on to infinity." His proof is essentially the one that we often show our students today. He notes that 1/4 plus 1/3 is greater than 1/2; the sum of the terms from 1/5 to 1/8 is greater than 1/2; and so on, obtaining infinitely many parts "of which any one will be greater than one-half foot and the whole will be infinite."

Oresme's result was rediscovered by Pietro Mengoli in 1672 and Jacques Bernoulli in 1689 [**1**]. The proof given by Bernoulli in *Ars conjectandi* has been characterized by William Dunham [**3**] as "entirely different, yet equally ingenious." According to the *Oxford English Dictionary*, the name "harmonic series" first appeared in *Chambers Cyclopedia* (1727–51): "Harmonical series is a series of many numbers in continual harmonical proportion."

The roots of the harmonic series can be traced back to some of the earliest mathematical traditions in Western culture. Based on the harmonic mean, it is related to harmonics in both music and geometry. This concept deserves its place as a fundamental idea of mathematical thought.

References

1. Carl B. Boyer, *A History of Mathematics*, Wiley, 1968, pp. 61, 406.
2. Richard L. Crocker, Pythagorean mathematics and music, *Journal of Aesthetics and Art Criticism*, 22 (1963–64), 189–198 and 325–335.
3. William Dunham, The Bernoullis and the harmonic series, *College Mathematics Journal*, 18 (1987), 18–23.
4. Joseph L. Ercolano, Remarks on the neglected mean, *Mathematics Teacher*, 67 (1973), 253–255.
5. Howard Eves, *A Survey of Geometry*, Revised Ed., Allyn & Bacon, 1972, p. 83.
6. Edward Grant, ed., *A Source Book in Medieval Science*, Harvard University Press, 1974, p. 135.
7. Thomas L. Heath, *A History of Greek Mathematics* (vol. 1), Dover reprint 1981, p. 85.
8. Alfred S. Posamentier, The neglected mean, *School Science and Mathematics*, 77 (1977), 339–344.
9. B. L. van der Waerden, *Science Awakening*, Wiley, 1963, pp. 149–159.

Factoring Quadratics

Stephen Kaczkowski (Stephen_kaczkowski@mail.com). Orange County Community College, Middletown NY 10940

Everyone knows that

$$x^2 + 5x + 6 = (x + 2)(x + 3) \quad \text{and}$$
$$x^2 + 5x - 6 = (x - 1)(x + 6).$$

Such pairs are not common, and not everyone is as familiar with

$$x^2 + 10x + 24 = (x + 4)(x + 6) \qquad x^2 + 13x + 30 = (x + 3)(x + 10)$$
$$x^2 + 10x - 24 = (x - 2)(x + 12) \qquad x^2 + 13x - 30 = (x - 2)(x + 15).$$

Here is how to find any number of such examples:

$$x^2 + Mx + N = (x + rt(s + t))(x + rs(s - t))$$
$$x^2 + Mx - N = (x - rt(s - t))(x + rs(s + t))$$

where r is any positive integer and s and t are relatively prime with opposite parity.