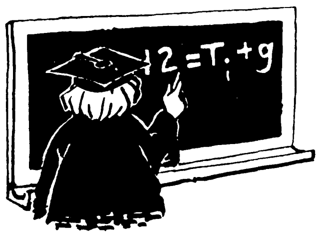

CLASSROOM CAPSULES

Edited by
Warren Page

Classroom Capsules serves to convey new insights on familiar topics and to enhance pedagogy through shared teaching experiences. Its format consists primarily of readily understood mathematics capsules which make their impact quickly and effectively. Such tidbits should be nurtured, cultivated, and presented for the benefit of your colleagues elsewhere. Queries, when available, will round out the column and serve to open further dialog on specific items of reader concern.

Readers are invited to submit material for consideration to:

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Counterexamples to a Comparison Test for Alternating Series

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Students studying infinite series frequently and mistakenly extend the comparison test for positive term series to exercises involving alternating series. For example, the argument given for the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$$

is: $1/(n^2+1) \leqslant 1/n$ *for all* n, *and* $\sum_{n=1}^{\infty}(-1)^n/n$ *converges*. Therefore, the original series converges by the comparison test. The conclusion, of course, is true but the reasoning is faulty. Students are reinforced in this misconception because the exercises to which the comparison test is applied happen to be series which are absolutely convergent. No example or exercise, which they are likely to encounter in calculus texts, serves as a counterexample.

G. H. Hardy (*Pure Mathematics*, Cambridge University Press, 1967, pp. 377–378) gives an example of alternating series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n} \tag{1}$$

that diverges, even though the positive terms $a_n = 1/(\sqrt{n}+(-1)^n)$ tend to zero as $n \to \infty$. The divergence of (1) can be readily established by rewriting it as

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+(-1)^n} = \sum_{n=2}^{\infty} \left\{ \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n+(-1)^n\sqrt{n}} \right\}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} - \sum_{n=2}^{\infty} \frac{1}{n+(-1)^n\sqrt{n}}, \tag{2}$$

and observing that the first series of (2) converges, whereas the second series diverges because $1/(n+(-1)^n\sqrt{n}) \geqslant 1/(2n)$ for each $n \geqslant 2$.

Hardy uses this example to emphasize the need for a_n to be strictly decreasing to zero as part of the hypothesis for the convergence of an alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$. But this same example can also be used as a counterexample to the extension of the comparison test to alternating series. Indeed, Hardy's divergent series (1) is "dominated" by the convergent alternating series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}-1},$$

as $1/(\sqrt{n}+(-1)^n) \leqslant 1/(\sqrt{n}-1)$ for all n. Thus, there is no "comparison test" for alternating series.

An example such as this should be presented to all students who study infinite series. Other such illustrations can also be presented—as, for example,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2+(-1)^n)},$$

a divergent alternating series that is dominated by the alternating harmonic series.

Editor's Note: Readers who are interested in this theme may want to refer to R. Lariviere's article "On a Convergence Test for Alternating Series," *Mathematics Magazine* 29 (November-December 1955) 88.

A Note on Differentiation

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The following technique illustrates an alternate method for deriving the product rule for differentiation.

If $f^2(x)$ is a differentiable function of x, then

$$[f^2(x)]' = \lim_{h\to 0} \frac{f^2(x+h)-f^2(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h)+f(x)][f(x+h)-f(x)]}{h}$$

$$= 2f(x)f'(x). \tag{*}$$